Linear Algebra (part 3): Eigenvalues and Eigenvectors (by Evan Dummit, 2016, v. 2.00)

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4 Eigenvalues and Eigenvectors

- We have discussed quite extensively the correspondence between solving a system of homogeneous linear equations and solving the matrix equation $A\mathbf{x} = \mathbf{0}$, for A an $n \times n$ matrix and $\mathbf{x}$ and $\mathbf{0}$ each $n \times 1$ column vectors.
- For reasons that will become more apparent soon, a more general version of this question which is also of interest is to solve the matrix equation $A\mathbf{x} = \lambda\mathbf{x}$, where λ is a scalar. (The original "homogeneous system" problem corresponds to $\lambda = 0$.)
- In the language of linear transformations, this says the following: given a linear transformation $T : V \to V$ from a vector space V to itself, on what vectors $\mathbf{x}$ does T act as multiplication by a constant λ?

4.1 Eigenvalues, Eigenvectors, Characteristic Polynomials

- Definition: For A an $n \times n$ matrix, a nonzero vector $\mathbf{x}$ with $A\mathbf{x} = \lambda\mathbf{x}$ is called an eigenvector of A, and the corresponding scalar λ is called an eigenvalue of A.
  - Important note: We do not consider the zero vector $\mathbf{0}$ an eigenvector.
  - Example: If $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 5, because $A\mathbf{x} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\mathbf{x}$.
  - Example: If $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, the vector $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 1, because $A\mathbf{x} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\mathbf{x}$.
  - Example: If $A = \begin{bmatrix} 2 & -4 & 5 \\ 2 & -2 & 5 \\ 2 & 1 & 2 \end{bmatrix}$, the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is an eigenvector of A with eigenvalue 4, because $A\mathbf{x} = \begin{bmatrix} 2 & -4 & 5 \\ 2 & -2 & 5 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix} = 4\mathbf{x}$.
- Eigenvalues and eigenvectors can also be complex numbers, even if the matrix A only has real-number entries.
  - Example: If $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$, the vector $\mathbf{x} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue $i = \sqrt{-1}$, because $A\mathbf{x} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} -1+2i \\ i \end{bmatrix} = i\mathbf{x}$.

- Example: If $A = \begin{bmatrix} 6 & 3 & -2 \\ -2 & 0 & 0 \\ 6 & 4 & 2 \end{bmatrix}$, the vector $\mathbf{x} = \begin{bmatrix} 1-i \\ 2i \\ 2 \end{bmatrix}$ is an eigenvector of A with eigenvalue $1+i$, because $A\mathbf{x} = \begin{bmatrix} 6 & 3 & -2 \\ -2 & 0 & 0 \\ 6 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1-i \\ 2i \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2+2i \\ 2+2i \end{bmatrix} = (1+i)\mathbf{x}$.

- It may at first seem that a given matrix may have many eigenvectors with many different eigenvalues. But in fact, any $n \times n$ matrix can only have a few eigenvalues, and there is a simple way to find them all using determinants:

- Proposition (Finding Eigenvalues): If A is an $n \times n$ matrix, the real or complex number λ is an eigenvalue of A if and only if $\det(\lambda I - A) = 0$.
  - Proof: Suppose λ is an eigenvalue with associated nonzero eigenvector $\mathbf{x}$: this is equivalent to saying $A\mathbf{x} = \lambda\mathbf{x}$.
  - Next observe that $\lambda\mathbf{x} = (\lambda I)\mathbf{x}$ where I is the $n \times n$ identity matrix.
  - Therefore, we can rewrite the eigenvalue equation $A\mathbf{x} = \lambda\mathbf{x} = (\lambda I)\mathbf{x}$ as $(\lambda I - A)\mathbf{x} = \mathbf{0}$.
  - But from our study of homogeneous systems of linear equations, the matrix equation $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nonzero solution for $\mathbf{x}$ if and only if the matrix $(\lambda I - A)$ is not invertible, which is in turn equivalent to saying that $\det(\lambda I - A) = 0$.

- When we expand the determinant $\det(tI - A)$, we will obtain a polynomial of degree n in the variable t.

- Definition: For an $n \times n$ matrix A, the degree-n polynomial $p(t) = \det(tI - A)$ is called the characteristic polynomial of A, and its roots are precisely the eigenvalues of A.
  - Some authors instead define the characteristic polynomial as the determinant of the matrix $A - tI$ rather than $tI - A$. We define it this way because then the coefficient of t^n will always be 1, rather than $(-1)^n$.

- When searching for roots of polynomials of small degree, the following case of the rational root test is often helpful.

- Proposition: Suppose the polynomial $p(t) = t^n + \cdots + b$ has integer coefficients and leading coefficient 1. Then any rational root of $p(t)$ must be an integer that divides b.
  - The proposition cuts down on the amount of trial and error necessary for finding rational roots of polynomials, since we only need to consider integers that divide the constant term.
  - Of course, a generic polynomial will not have a rational root, so to compute eigenvalues in practice one generally needs to use numerical approximations. (But we will arrange the examples so that the polynomials will factor nicely.)

- Example: Find the eigenvalues of $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$.
  - First we compute the characteristic polynomial $\det(tI - A) = \begin{vmatrix} t-3 & -1 \\ -2 & t-4 \end{vmatrix} = t^2 - 7t + 10$.
  - The eigenvalues are then the zeroes of this polynomial. Since $t^2 - 7t + 10 = (t-2)(t-5)$ we see that the zeroes are $t = 2$ and $t = 5$, meaning that the eigenvalues are $\boxed{2 \text{ and } 5}$.

- Example: Find the eigenvalues of $A = \begin{bmatrix} 1 & 4 & \sqrt{3} \\ 0 & 3 & -8 \\ 0 & 0 & \pi \end{bmatrix}$.
  - Observe that $\det(tI - A) = \begin{vmatrix} t-1 & -4 & -\sqrt{3} \\ 0 & t-3 & 8 \\ 0 & 0 & t-\pi \end{vmatrix} = (t-1)(t-3)(t-\pi)$ since the matrix is upper-triangular. Thus, the eigenvalues are $\boxed{1, 3, \pi}$.

- The idea from the example above works in generality:
- Proposition (Eigenvalues of Triangular Matrix): The eigenvalues of an upper-triangular matrix or of a lower-triangular matrix are its diagonal entries.
  - Proof: If A is an $n \times n$ upper-triangular (or lower-triangular) matrix, then so is $tI - A$.
  - Then by properties of determinants, $\det(tI - A)$ is equal to the product of the diagonal entries of $tI - A$.
  - Since these diagonal entries are simply $t - a_{i,i}$ for $1 \le i \le n$, the eigenvalues are $a_{i,i}$ for $1 \le i \le n$, which are simply the diagonal entries of A.
- It can happen that the characteristic polynomial has a repeated root. In such cases, it is customary to note that the associated eigenvalue has "multiplicity" and include the eigenvalue the appropriate number of extra times when listing them.
  - For example, if a matrix has characteristic polynomial $t^2(t-1)^3$, we would say the eigenvalues are 0 with multiplicity 2, and 1 with multiplicity 3. We would list the eigenvalues as $\lambda = 0, 0, 1, 1, 1$.
- Example: Find the eigenvalues of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
  - By expanding along the bottom row we see $\det(tI - A) = \begin{vmatrix} t-1 & 1 & 0 \\ -1 & t-3 & 0 \\ 0 & 0 & t \end{vmatrix} = t \begin{vmatrix} t-1 & 1 \\ -1 & t-3 \end{vmatrix} = t(t^2 - 4t + 4)$.
  - Since $t^2 - 4t + 4 = (t-2)^2$ we see that the characteristic polynomial has a single root $t = 0$ and a double root $t = 2$.
  - Thus, A has an eigenvalue 0 of multiplicity 1 and an eigenvalue 2 of multiplicity 2. As a list, the eigenvalues are $\lambda = \boxed{0,\ 2,\ 2}$.
- Example: Find the eigenvalues of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.
  - By expanding along the top row,

$$\begin{aligned} \det(tI - A) &= \begin{vmatrix} t-1 & -1 & 0 \\ 0 & t-1 & -1 \\ 0 & 0 & t-1 \end{vmatrix} \\ &= (t-1) \begin{vmatrix} t-1 & -1 \\ 0 & t-1 \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 \\ 0 & t-1 \end{vmatrix} \\ &= (t-1)(t-1)^2 = (t-1)^3. \end{aligned}$$

  - Thus, the characteristic polynomial has a triple root $t = 1$.
  - Thus, A has an eigenvalue 1 of multiplicity 3. As a list, the eigenvalues are $\lambda = \boxed{1,\ 1,\ 1}$.
- Note also that the characteristic polynomial may have non-real numbers as roots.
  - As we saw above, matrices with real entries may have non-real eigenvalues. Such non-real eigenvalues are absolutely acceptable: the only wrinkle is that the eigenvectors for these eigenvalues will also necessarily contain non-real entries.
  - If A has real number entries, then because the characteristic polynomial of A is a polynomial with real coefficients, any non-real roots of the characteristic polynomial will come in complex conjugate pairs.
- Example: Find the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$.

- First we compute the characteristic polynomial $\det(tI - A) = \begin{vmatrix} t-1 & -1 \\ 2 & t-3 \end{vmatrix} = t^2 - 4t + 5$.
- The eigenvalues are then the zeroes of this polynomial. By the quadratic formula, the roots are $\dfrac{4 \pm \sqrt{-4}}{2} = 2 \pm i$, so the eigenvalues are $\boxed{2+i \text{ and } 2-i}$.

- Example: Find the eigenvalues of $A = \begin{bmatrix} -1 & 2 & -4 \\ 3 & -2 & 1 \\ 4 & -4 & 4 \end{bmatrix}$.
  - By expanding along the top row,

$$\begin{aligned} \det(tI - A) &= \begin{vmatrix} t+1 & -2 & 4 \\ -3 & t+2 & -1 \\ -4 & 4 & t-4 \end{vmatrix} \\ &= (t+1)\begin{vmatrix} t+2 & -1 \\ 4 & t-4 \end{vmatrix} + 2\begin{vmatrix} -3 & -1 \\ -4 & t-4 \end{vmatrix} + 4\begin{vmatrix} -3 & t+2 \\ -4 & 4 \end{vmatrix} \\ &= (t+1)(t^2 - 2t - 4) + 2(-3t + 8) + 4(4t - 4) \\ &= t^3 - t^2 + 4t - 4. \end{aligned}$$

  - To find the roots, we wish to solve the cubic equation $t^3 - t^2 + 4t - 4 = 0$.
  - By the rational root test, if the polynomial has a rational root then it must be an integer dividing -4: that is, one of ± 1, ± 2, ± 4. Testing the possibilities reveals that $t = 1$ is a root, and then we get the factorization $(t-1)(t^2+4) = 0$.
  - The roots of the quadratic are $t = \pm 2i$, so the eigenvalues are $\boxed{1,\ 2i,\ -2i}$.

4.2 Eigenspaces

- Using the characteristic polynomial, we can find all the eigenvalues of a matrix A without actually determining the associated eigenvectors. However, we often also want to find the eigenvectors associated to each eigenvalue.
- We might hope that there is a straightforward way to describe all the eigenvectors, and (conveniently) there is: the set of all eigenvectors with a particular eigenvalue λ has a vector space structure.
- Proposition: For a fixed value of λ, the set S_λ whose elements are the eigenvectors $\mathbf{x}$ with $A\mathbf{x} = \lambda\mathbf{x}$, together with the zero vector, is a subspace of $V = \mathbb{R}^n$ (thought of as $n \times 1$ column vectors). This set S_λ is called the eigenspace associated to the eigenvalue λ, or the λ-eigenspace.
  - Proof: Notice that because we explicitly included the zero vector, S_λ is simply the set of all vectors such that $A\mathbf{v} = \lambda\mathbf{v}$. Now we simply check the subspace criterion:
  - [S1]: S_λ contains the zero vector.
  - [S2]: S_λ is closed under addition, because if $A\mathbf{x}_1 = \lambda\mathbf{x}_1$ and $A\mathbf{x}_2 = \lambda\mathbf{x}_2$, then $A(\mathbf{x}_1 + \mathbf{x}_2) = \lambda(\mathbf{x}_1 + \mathbf{x}_2)$.
  - [S3]: S_λ is closed under scalar multiplication, because if $A\mathbf{x} = \lambda\mathbf{x}$, then for any scalar β, $A(\beta\mathbf{x}) = \beta(A\mathbf{x}) = \beta(\lambda\mathbf{x}) = \lambda(\beta\mathbf{x})$.
- Example: Find the 1-eigenspaces, and their dimensions, for $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
  - For the 1-eigenspace of A, we want to find all vectors with $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$.
  - Clearly, all vectors satisfy this equation, so the 1-eigenspace of A is the set of all vectors $\boxed{\begin{bmatrix} a \\ b \end{bmatrix}}$, and has dimension 2.

- For the 1-eigenspace of B, we want to find all vectors with $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$, or equivalently, $\begin{bmatrix} a+b \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$.
- The vectors satisfying the equation are those with $b = 0$, so the 1-eigenspace of B is the set of vectors of the form $\boxed{\begin{bmatrix} a \\ 0 \end{bmatrix}}$, and has dimension 1.
- Notice that the characteristic polynomial of each matrix is $(t-1)^2$, since both matrices are upper-triangular, and they both have a single eigenvalue $\lambda = 1$ of multiplicity 2. Nonetheless, the matrices do not have the same eigenvectors, and the dimensions of their 1-eigenspaces are different.

- Now, since the λ-eigenspace is a vector space, if we want to describe all eigenvectors for a given eigenvalue λ, we can simply find a basis for the λ-eigenspace.
  - For each eigenvalue λ, our goal is to solve for all vectors $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$.
  - Equivalently, we wish to find the vectors $\mathbf{x}$ satisfying the matrix equation $(\lambda I - A)\mathbf{x} = \mathbf{0}$, which (per our analysis of systems of linear equations) can be done by row-reducing the matrix $\lambda I - A$. We have also described the procedure for extracting a basis for the solution set.
  - The resulting solution vectors $\mathbf{x}$ form the eigenspace associated to λ, and the nonzero vectors in the space are the eigenvectors corresponding to λ.

- To find all the eigenvalues and eigenvectors of a matrix A, follow these steps:
  - Step 1: Write down the matrix $tI - A$ and compute its determinant (using any method) to obtain the characteristic polynomial $p(t)$.
  - Step 2: Set $p(t)$ equal to zero and solve. The roots are precisely the eigenvalues λ of A.
  - Step 3: For each eigenvalue λ, solve for all vectors $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$: this is the set of solutions to $(\lambda I - A)\mathbf{x} = \mathbf{0}$, which is equivalent to the nullspace of $\lambda I - A$ and may be computed by row-reduction.

- Example: Find all eigenvalues, and a basis for each eigenspace, for the matrix $A = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$.
  - We have $tI - A = \begin{bmatrix} t-2 & -2 \\ -3 & t-1 \end{bmatrix}$, so $p(t) = \det(tI - A) = (t-2)(t-1) - (-2)(-3) = t^2 - 3t - 4$.
  - Since $p(t) = t^2 - 3t - 4 = (t-4)(t+1)$, the eigenvalues are $\boxed{\lambda = -1, 4}$.
  - For $\lambda = -1$, we want to find the nullspace of $\begin{bmatrix} -1-2 & -2 \\ -3 & -1-1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -3 & -2 \end{bmatrix}$. By row-reducing we find the row-echelon form is $\begin{bmatrix} -3 & -2 \\ 0 & 0 \end{bmatrix}$, so the nullspace is 1-dimensional and is spanned by $\boxed{\begin{bmatrix} -2 \\ 3 \end{bmatrix}}$.
  - For $\lambda = 4$, we want to find the nullspace of $\begin{bmatrix} 4-2 & -2 \\ -3 & 4-1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}$. By row-reducing we find the row-echelon form is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, so the nullspace is 1-dimensional and is spanned by $\boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$.

- Example: Find all eigenvalues, and a basis for each eigenspace, for the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 3 \\ -1 & 0 & 3 \end{bmatrix}$.
  - First, we have $tI - A = \begin{bmatrix} t-1 & 0 & -1 \\ 1 & t-1 & -3 \\ 1 & 0 & t-3 \end{bmatrix}$, so $p(t) = (t-1) \cdot \begin{vmatrix} t-1 & -3 \\ 0 & t-3 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & t-1 \\ 1 & 0 \end{vmatrix} =$ $(t-1)^2(t-3) + (t-1)$.

- Since $p(t) = (t-1) \cdot [(t-1)(t-3)+1] = (t-1)(t-2)^2$, the eigenvalues are $\boxed{\lambda = 1, 2, 2}$.
- For $\lambda = 1$ we want to find the nullspace of $\begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 1-1 & -3 \\ 1 & 0 & 1-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 1 & 0 & -3 \end{bmatrix}$. This matrix's reduced row-echelon form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so the nullspace is 1-dimensional and spanned by $\boxed{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$.
- For $\lambda = 2$ we want to find the nullspace of $\begin{bmatrix} 2-1 & 0 & -1 \\ 1 & 2-1 & -3 \\ 1 & 0 & 2-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ 1 & 0 & -1 \end{bmatrix}$. This matrix's reduced row-echelon form is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, so the nullspace is 1-dimensional and spanned by $\boxed{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}$.

- Example: Find all eigenvalues, and a basis for each eigenspace, for the matrix $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.
  - We have $tI - A = \begin{bmatrix} t & 0 & 0 \\ -1 & t & 1 \\ 0 & -1 & t \end{bmatrix}$, so $p(t) = \det(tI - A) = t \cdot \begin{vmatrix} t & 1 \\ -1 & t \end{vmatrix} = t \cdot (t^2+1)$.
  - Since $p(t) = t \cdot (t^2+1)$, the eigenvalues are $\boxed{\lambda = 0, i, -i}$.
  - For $\lambda = 0$ we want to find the nullspace of $\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. This matrix's reduced row-echelon form is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so the nullspace is 1-dimensional and spanned by $\boxed{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$.
  - For $\lambda = i$ we want to find the nullspace of $\begin{bmatrix} i & 0 & 0 \\ -1 & i & 1 \\ 0 & -1 & i \end{bmatrix}$. This matrix's reduced row-echelon form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix}$, so the nullspace is 1-dimensional and spanned by $\boxed{\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}}$.
  - For $\lambda = -i$ we want to find the nullspace of $\begin{bmatrix} -i & 0 & 0 \\ -1 & -i & 1 \\ 0 & -1 & -i \end{bmatrix}$. This matrix's reduced row-echelon form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix}$, so the nullspace is 1-dimensional and spanned by $\boxed{\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}}$.

- Notice that in the example above, with a real matrix having complex-conjugate eigenvalues, the associated eigenvectors were also complex conjugates. This is no accident:

- Proposition (Conjugate Eigenvalues): If A is a real matrix and $\mathbf{v}$ is an eigenvector with a complex eigenvalue λ, then the complex conjugate $\overline{\mathbf{v}}$ is an eigenvector with eigenvalue $\overline{\lambda}$. In particular, a basis for the $\overline{\lambda}$-eigenspace is given by the set of complex conjugates of a basis for the λ-eigenspace.
  - Proof: The first statement follows from the observation that the complex conjugate of a product or sum is the appropriate product or sum of complex conjugates, so if A and B are any matrices of compatible sizes for multiplication, we have $\overline{A \cdot B} = \overline{A} \cdot \overline{B}$.

- Thus, if $A\mathbf{v} = \lambda\mathbf{v}$, taking complex conjugates gives $\overline{A}\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$, and since $\overline{A} = A$ because A is a real matrix, we see $A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$: thus, $\overline{\mathbf{v}}$ is an eigenvector with eigenvalue $\overline{\lambda}$.
- The second statement follows from the first, since complex conjugation does not affect linear independence or dimension.

• Example: Find all eigenvalues, and a basis for each eigenspace, for the matrix $A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$.

- We have $tI - A = \begin{bmatrix} t-3 & 1 \\ -2 & t-5 \end{bmatrix}$, so $p(t) = \det(tI - A) = (t-3)(t-5) - (-2)(1) = t^2 - 8t + 17$.
- Using the quadratic equation yields that the eigenvalues are $\boxed{\lambda = 4 \pm i}$.
- For $\lambda = 4+i$, we want to find the nullspace of $\begin{bmatrix} t-3 & 1 \\ -2 & t-5 \end{bmatrix} = \begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix}$. Row-reducing this matrix yields
$$\begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix} \xrightarrow{R_2+(1-i)R_1} \begin{bmatrix} 1+i & 1 \\ 0 & 0 \end{bmatrix}$$
from which we can see that the eigenspace is 1-dimensional and spanned by $\boxed{\begin{bmatrix} 1 \\ -1-i \end{bmatrix}}$.
- For $\lambda = 4-i$ we can simply take the conjugate of the calculation we made for $\lambda = 4+i$: thus, the $(4-i)$-eigenspace is also 1-dimensional and spanned by $\boxed{\begin{bmatrix} 1 \\ -1+i \end{bmatrix}}$.

• Example: Find all eigenvalues, and a basis for each eigenspace, for the matrix $A = \begin{bmatrix} 5 & -4 & -6 \\ 2 & 1 & -2 \\ 2 & -3 & -3 \end{bmatrix}$.

- We have $tI - A = \begin{bmatrix} t-5 & 4 & 6 \\ -2 & t-1 & 2 \\ -2 & 3 & t+3 \end{bmatrix}$, so $\det(tI-A) = (t-5)(t^2+2t-9) - 4(-2t-2) + 6(2t-8) = t^3 - 3t^2 + t + 5$.
- Searching for small rational roots produces the root $t = -1$, and factoring yields $t^3 - 3t^2 + t + 5 = (t+1)(t^2 - 4t + 5)$. The roots of the quadratic are $2 \pm i$, so $\boxed{\lambda = -1,\, 2+i,\, 2-i}$.
- For $\lambda = -1$ we want to find the nullspace of $\begin{bmatrix} \lambda-5 & 4 & 6 \\ -2 & \lambda-1 & 2 \\ -2 & 3 & \lambda+3 \end{bmatrix} = \begin{bmatrix} -6 & 4 & 6 \\ -2 & -2 & 2 \\ -2 & 3 & 2 \end{bmatrix}$. This matrix's reduced row-echelon form is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so the nullspace is 1-dimensional and spanned by $\boxed{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$.
- For $\lambda = 2+i$ we want to find the nullspace of $\begin{bmatrix} \lambda-5 & 4 & 6 \\ -2 & \lambda-1 & 2 \\ -2 & 3 & \lambda+3 \end{bmatrix} = \begin{bmatrix} -3+i & 4 & 6 \\ -2 & 1+i & 2 \\ -2 & 3 & 5+i \end{bmatrix}$.

Row-reducing this matrix yields
$$\begin{bmatrix} -3+i & 4 & 6 \\ -2 & 1+i & 2 \\ -2 & 3 & 5+i \end{bmatrix} \xrightarrow{\frac{-1+i}{2}R_2} \begin{bmatrix} -3+i & 4 & 6 \\ 1-i & -1 & -1+i \\ -2 & 3 & 5+i \end{bmatrix} \xrightarrow[R_3+(1+i)R_2]{R_1+(2+i)R_2} \begin{bmatrix} 0 & 2-i & 3+i \\ 1-i & -1 & -1+i \\ 0 & 2-i & 3+i \end{bmatrix}$$
$$\xrightarrow{R_1-R_3} \begin{bmatrix} 0 & 2-i & 3+i \\ 1-i & -1 & -1+i \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{2+i}{5}R_1} \begin{bmatrix} 0 & 1 & 1+i \\ 1-i & -1 & -1+i \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 0 & 1 & 1+i \\ 1-i & 0 & 2i \\ 0 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow{\frac{1+i}{2}R_2} \begin{bmatrix} 0 & 1 & 1+i \\ 1 & 0 & -1+i \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & -1+i \\ 0 & 1 & 1+i \\ 0 & 0 & 0 \end{bmatrix}$$

from which we see that the nullspace is 1-dimensional and spanned by $\boxed{\begin{bmatrix} 1-i \\ -1-i \\ 1 \end{bmatrix}}$.

- For $\lambda = 2-i$ we can simply take the conjugate of the calculation we made for $\lambda = 2+i$: thus, the $(2-i)$-eigenspace is also 1-dimensional and spanned by $\boxed{\begin{bmatrix} 1+i \\ -1+i \\ 1 \end{bmatrix}}$.

4.3 Additional Properties of Eigenvalues

- We will now mention a few useful theoretical results about eigenvalues, eigenvectors, and eigenspaces.
- Theorem (Eigenvalue Multiplicity): If λ is an eigenvalue of the matrix A which appears exactly k times as a root of the characteristic polynomial, then the dimension of the eigenspace corresponding to λ is at least 1 and at most k.
  - Remark: The number of times that λ appears as a root of the characteristic polynomial is sometimes called the "algebraic multiplicity" of λ, and the dimension of the eigenspace corresponding to λ is sometimes called the "geometric multiplicity" of λ. In this language, the theorem above says that the geometric multiplicity is less than or equal to the algebraic multiplicity.
  - Example: If the characteristic polynomial of a matrix is $(t-1)^3(t-3)^2$, then the eigenspace for $\lambda = 1$ is at most 3-dimensional, and the eigenspace for $\lambda = 3$ is at most 2-dimensional.
  - Proof: The statement that the eigenspace has dimension at least 1 is immediate, because (by assumption) λ is a root of the characteristic polynomial and therefore has at least one nonzero eigenvector associated to it.
  - For the other statement, observe that the dimension of the λ-eigenspace is the dimension of the solution space of the homogeneous system $(\lambda I - A) \cdot \mathbf{x} = \mathbf{0}$. (Equivalently, it is the dimension of the nullspace of the matrix $\lambda I - A$.)
  - If λ appears k times as a root of the characteristic polynomial, then when we put the matrix $\lambda I - A$ into its reduced row-echelon form B, we claim that B must have at most k rows of all zeroes.
  - Otherwise, the matrix B (and hence $\lambda I - A$ too, since the nullity and rank of a matrix are not changed by row operations) would have 0 as an eigenvalue more than k times, because B is in echelon form and therefore upper-triangular.
  - But the number of rows of all zeroes in a square matrix in reduced row-echelon form is the same as the number of nonpivotal columns, which is the number of free variables, which is the dimension of the solution space.
  - So, putting all the statements together, we see that the dimension of the eigenspace is at most k.
- Theorem (Independent Eigenvectors): If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are eigenvectors of A associated to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.
  - Proof: Suppose we had a nontrivial dependence relation between $\mathbf{v}_1, \ldots, \mathbf{v}_n$, say $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$. (Note that at least two coefficients have to be nonzero, because none of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the zero vector.)
  - Multiply both sides by the matrix A: this gives $A \cdot (a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = A \cdot \mathbf{0} = \mathbf{0}$.
  - Now since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are eigenvectors this says $a_1(\lambda_1\mathbf{v}_1) + \cdots + a_n(\lambda_n\mathbf{v}_n) = \mathbf{0}$.
  - But now if we scale the original equation by λ_1 and subtract (to eliminate $\mathbf{v}_1$), we obtain $a_2(\lambda_2 - \lambda_1)\mathbf{v}_2 + a_3(\lambda_3 - \lambda_1)\mathbf{v}_3 + \cdots + a_n(\lambda_n - \lambda_1)\mathbf{v}_n = \mathbf{0}$.
  - Since by assumption all of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ were different, this dependence is still nontrivial, since each of $\lambda_j - \lambda_1$ is nonzero, and at least one of $a_2, \cdots, a_n$ is nonzero.
  - But now we can repeat the process to eliminate each of $\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_{n-1}$ in turn. Eventually we are left with the equation $b\mathbf{v}_n = \mathbf{0}$ for some nonzero b. But this is impossible, because it would say that $\mathbf{v}_n = \mathbf{0}$, contradicting our definition saying that the zero vector is not an eigenvector.

- So there cannot be a nontrivial dependence relation, meaning that $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent.

- Corollary: If A is an $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are (any) eigenvectors associated to those respective eigenvalues, then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ form a basis for $\mathbb{R}^n$.
  - This result follows from the previous theorem: it guarantees that the n vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent, so they must be a basis of the n-dimensional vector space $\mathbb{R}^n$.

- Theorem (Eigenvalues, Trace, and Determinant): The product of the eigenvalues of A is the determinant of A, and the sum of the eigenvalues of A equals the trace of A.
  - Recall that the trace of a matrix is defined to be the sum of its diagonal entries.
  - Proof: Let $p(t)$ be the characteristic polynomial of A.
  - If we expand out the product $p(t) = (t - \lambda_1) \cdot (t - \lambda_2) \cdots (t - \lambda_n)$, we see that the constant term is equal to $(-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$.
  - But the constant term is also just $p(0)$, and since $p(t) = \det(tI - A)$ we have $p(0) = \det(-A) = (-1)^n \det(A)$: thus, $\lambda_1 \lambda_2 \cdots \lambda_n = \det(A)$.
  - Furthermore, upon expanding out the product $p(t) = (t - \lambda_1) \cdot (t - \lambda_2) \cdots (t - \lambda_n)$, we see that the coefficient of t^{n-1} is equal to $-(\lambda_1 + \cdots + \lambda_n)$.
  - If we expand out the determinant $\det(tI - A)$ to find the coefficient of t^{n-1}, we can show (with a little bit of effort) that the coefficient is the negative of the sum of the diagonal entries of A.
  - Thus, setting the two expressions equal shows that the sum of the eigenvalues equals the trace of A.

- Example: Find the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ -2 & -1 & -2 \\ 2 & 2 & -3 \end{bmatrix}$, and verify the formulas for trace and determinant in terms of the eigenvalues.
  - By expanding along the top row, we can compute

$$\begin{aligned} \det(tI - A) &= (t-2) \begin{vmatrix} t+1 & 2 \\ -2 & t+3 \end{vmatrix} - (-1) \begin{vmatrix} 2 & 2 \\ -2 & t+3 \end{vmatrix} + (-1) \begin{vmatrix} 2 & t+1 \\ -2 & -2 \end{vmatrix} \\ &= (t-2)(t^2 + 4t + 7) + (2t + 10) - (2t - 2) = t^3 + 2t^2 - t - 2. \end{aligned}$$

  - To find the eigenvalues, we wish to solve the cubic equation $t^3 + 2t^2 - t - 2 = 0$.
  - By the rational root test, if the polynomial has a rational root then it must be an integer dividing -2: that is, one of ± 1, ± 2. Testing the possibilities reveals that $t = 1$, $t = -1$, and $t = -2$ are each roots, from which we obtain the factorization $(t-1)(t+1)(t+2) = 0$.
  - Thus, the eigenvalues are $t = -2, -1, 1$.
  - We see that $\text{tr}(A) = 2 + (-1) + (-3) = -2$, while the sum of the eigenvalues is $(-2) + (-1) + 1 = -2$. They are indeed equal.
  - For the determinant, we compute

$$\begin{aligned} \det(A) &= 2 \begin{vmatrix} -1 & -2 \\ 2 & -3 \end{vmatrix} - 1 \begin{vmatrix} -2 & -2 \\ 2 & -3 \end{vmatrix} + 1 \begin{vmatrix} -2 & -1 \\ 2 & 2 \end{vmatrix} \\ &= 2(7) - 1(10) + 1(-2) = 2. \end{aligned}$$

  The product of the eigenvalues is $(-2)(-1)(1) = 2$, so the result holds as claimed.

Well, you're at the end of my handout. Hope it was helpful.