Linear Algebra (part 0) : Preliminaries (by Evan Dummit, 2019, v. 2.10)

Contents

0 Preliminaries

In this chapter we will review a few preliminary notions that are necessary for our formal treatment of linear algebra in subsequent chapters. We will briefly discuss properties of vectors in $\mathbb{R}^n$, matrices and systems of linear equations, determinants, complex numbers, fields, polynomials, and mathematical induction. Some of the material (on vectors, matrices, and polynomials) will be used as motivation for our axiomatic development of vector spaces, while other material (on determinants, fields, induction) will occasionally be needed during our later discussions.

0.1 Sets, Numbers, and Functions

- We very briefly recall a few basic notions about sets, numbers, and functions.
- Definition: A set is a well-defined collection of distinct elements.
  - The elements of a set can be essentially anything: integers, real numbers, other sets, people.
  - Sets are generally denoted by capital or script letters, and when listing the elements of a set, curly brackets $\{\cdot\}$ are used.
  - Sets do not have to have any elements: the empty set $\emptyset = \{\ \}$ is the set with no elements at all.
  - Two sets are the same precisely if all of their elements are the same. The elements in a set are not ordered, and no element can appear in a set more than once: thus the sets $\{1, 4\}$ and $\{4, 1\}$ are the same.
  - We write $x \in S$ when x is an element of the set S, and $x \notin S$ when x is not an element of the set S.
  - Example: For $S = \{1, 2, 5\}$ we have $1 \in S$ and $5 \in S$ but $3 \notin S$ and $\pi \notin S$.
- There are two primary ways to describe a set.
  - One way is to list all the elements: for example, $A = \{1, 2, 4, 5\}$ is the set containing the four numbers 1, 2, 4, and 5.

- The other way to define a set is to describe properties of its elements[1]: for example, the set S of one-letter words in the English alphabet has two elements: $S = \{a, I\}$.

- We often employ "set-builder" notation for sets: for example, the set S of real numbers between 0 and 5 is denoted $S = \{x : x \text{ is a real number and } 0 \leq x \leq 5\}$.
  - Some authors use a vertical pipe | instead of a colon : but this distinction is irrelevant.

- Definition: If A and B are two sets with the property that every element of A is also an element of B, we say A is a subset of B (or that A is contained in B) and write $A \subseteq B$. [2]
  - Example: If $A = \{1, 2, 3\}$, $B = \{1, 4, 5\}$, and $C = \{1, 2, 3, 4, 5\}$, then $A \subseteq C$ and $B \subseteq C$ but neither A nor B is a subset of the other.

- Definition: If A and B are two sets, then the intersection $A \cap B$ is the set of all elements contained in both A and B. The union $A \cup B$ is the set of all elements contained in either A or B (or both).
  - Example: If $A = \{1, 2, 3\}$ and $B = \{1, 4, 5\}$, then $A \cap B = \{1\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$.

- Certain sets of numbers are extremely important, and arise often.

- Definition: The positive integers (1, 2, 3, ...) are obtained by adding 1 to itself some number of times; the other integers are 0 and the negatives of the positive integers (−1,−2,−3, ...). Integers can be added, subtracted, multiplied, but only sometimes divided. The set of positive integers is denoted $\mathbb{N}$ ("naturals") and the set of all integers is denoted $\mathbb{Z}$ ("Zahlen", German for "numbers").
  - Examples of integers: 3, 0, −666, 1337, $10^{10^{10}}$.

- Definition: The rational numbers are numbers of the form a/b where a and b are integers and b is not zero. Rational numbers can be added, subtracted, multiplied, and always divided (except by 0). The set of rational numbers is denoted $\mathbb{Q}$ ("quotients").
  - Examples of rational numbers: $\frac{1}{2}$, $-\frac{225}{1037}$, 11, 0.
  - The basic operations are $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

- Definition: The real numbers are harder to define simply, but (roughly speaking) they are obtained by filling in the "gaps" between the rational numbers. A common way to think of real numbers is as (infinite) decimal sequences[3]. The set of real numbers is denoted $\mathbb{R}$ ("reals"). Real numbers can be added, subtracted, multiplied, and always divided (except by 0).
  - Examples of real numbers: π, $\sqrt{2}$, $\frac{\pi+\sqrt{2}}{5}$, 11, 0, 0.12131415161718192021....

- We can also speak of functions in this general context.

- Definition: A function is a relation between a set of inputs (called the domain of the function) and a set of outputs (called the range of the function): to each element of the domain, the function associates a single value in the range. Two functions are the same if they have the same domain and associate the same value in the range to each element of their common domain.
  - Example: Consider $f(x) = x^3$, with domain and range both the set of real numbers. This function f sends each real number x to its cube x^3: thus $f(2) = 8$, $f(0) = 0$, and $f(-1) = -1$.

[1] It is possible to run into trouble by trying to define sets in this "naive" way of specifying qualities of their elements. In general, one must be more careful when defining arbitrary sets, although we will not worry about this.

[2] Subset notation is not universally agreed-upon: the notation $A \subset B$ is also commonly used to say that A is a subset of B. The difference is not terribly relevant except for when A can be equal to B: some authors intend $A \subset B$ to include the possibility that A could be equal to B, while others insist that $A \subset B$ means that A is a subset of B which cannot be all of B.

[3] Some real numbers have two decimal sequences: 1.000... = 0.999... are two different ways of writing the positive integer 1. A similar ambiguity occurs with any other decimal number ending in an infinite string of 9s, but these are the only real numbers with two decimal expansions.

- If the function is f, we write $f : S \to T$ to indicate that S is the domain of f and that the range of f is contained in T.[4]
- Example: Consider the function $g : \{1, 2\} \to \{1, 2, 3\}$ with $g(1) = 2$ and $g(2) = 3$. The range of g is the set $\{2, 3\}$.
- In general, unless specified, the domain of a function is the largest possible set for which the definition of the function makes sense.

- The notation $f(g(x))$ symbolizes the result of applying f to the value $g(x)$: this is called function composition, and is well-defined provided that the range of g is a subset of the domain of f. We use the notation $f \circ g$ to refer to the composite function itself, so that $(f \circ g)(x) = f(g(x))$.
  - Example: For $g : \{1, 2\} \to \{2, 3\}$ having $g(1) = 3$ and $g(2) = 2$, and $f : \{2, 3\} \to \{3, 4\}$ having $f(2) = 3$ and $f(3) = 4$, the composition $f \circ g : \{1, 2\} \to \{3, 4\}$ has $(f \circ g)(1) = f(g(1)) = f(3) = 4$ and $(f \circ g)(2) = f(g(2)) = f(2) = 3$.
  - One can also compute other compositions, like $g \circ f$ or $f \circ f \circ f$. In general, it will be the case that $f \circ g$ and $g \circ f$ are *completely unrelated* functions! (In the example above, $g \circ f$ is not even defined.)

- Definition: A function f is one-to-one (or injective) if $f(a) = f(b)$ implies $a = b$, or equivalently, if $a \neq b$ implies $f(a) \neq f(b)$. In other words, f is one-to-one if unequal elements in the domain are sent to unequal elements in the range.
  - For functions whose domain and range are (subsets of) the real numbers, one-to-one functions satisfy the "horizontal line test": a horizontal line can intersect the graph of the function at most once.

- Definition: A one-to-one function $f(x)$ has an inverse function $f^{-1}(x)$ defined so that $f^{-1}(f(x)) = x$ for every x in the domain of f, and $f(f^{-1}(y)) = y$ for every y in the range of f.
  - To compute the inverse function of f, simply solve the equation $y = f(x)$ for y in terms of x: this will give $x = f^{-1}(y)$.

0.2 Vectors in $\mathbb{R}^n$

- A vector, as we typically think of it, is a quantity which has both a magnitude and a direction. This is in contrast to a scalar, which carries only a magnitude.
  - Real-valued vectors are extremely useful in just about every aspect of the physical sciences, since just about everything in Newtonian physics is a vector: position, velocity, acceleration, forces, etc. There is also "vector calculus" (namely, calculus in the context of vector fields) which is typically part of a multivariable calculus course; it has many applications to physics as well.

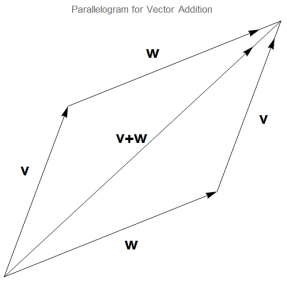
- We often think of vectors geometrically, as a directed line segment (having a starting point and an endpoint).
- Algebraically, we write a vector as an ordered tuple of coordinates: we denote the n-dimensional vector from the origin to the point $(a_1, a_2, \cdots, a_n)$ as $\mathbf{v} = \langle a_1, a_2, \cdots, a_n \rangle$, where the a_i are real-number scalars.
  - Some vectors: $\langle 1, 2 \rangle$, $\langle 3, 5, -1 \rangle$, $\left\langle -\pi, e^2, 27, 3, \frac{4}{3}, 0, 0, -1 \right\rangle$.
  - Notation: We use angle brackets $\langle \cdot \rangle$ rather than parentheses $(\cdot)$ so as to draw a visual distinction between a vector and the coordinates of a point in space. We also draw arrows above vectors (as $\vec{v}$) or typeset them in boldface (as $\mathbf{v}$) in order to set them apart from scalars. Boldface is hard to produce without a computer, so it is highly recommended to use the arrow notation $\vec{v}$ when writing by hand.

- We often think of, and draw, vectors as directed line segments.

[4] Technically, the set T in the notation $f : S \to T$ is called the codomain (or "target") of f, since the range of f is only the set of values of the form $f(x)$ for some x in the domain of f. For $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, the range of f is only the set $[0, \infty)$ and not all of $\mathbb{R}$.

- However, technically speaking, vectors are slightly different from directed segments, because we don't care where a vector starts: we only care about the difference between the starting and ending positions. Thus: the directed segment whose start is $(0,0)$ and end is $(1,1)$ and the segment starting at $(1,1)$ and ending at $(2,2)$ represent the *same vector* $\langle 1,1\rangle$.

- We can add vectors of the same type: if $\mathbf{v} = \langle a_1, \cdots, a_n\rangle$ and $\mathbf{w} = \langle b_1, \cdots, b_n\rangle$ then $\mathbf{v}+\mathbf{w} = \langle a_1+b_1, \cdots, a_n+b_n\rangle$.
  - Geometrically, we visualize vector addition using a parallelogram whose pairs of parallel sides are $\mathbf{v}$ and $\mathbf{w}$ and whose diagonal is $\mathbf{v}+\mathbf{w}$:



  - We can also scale a vector by a scalar, one component at a time: if r is a scalar, then $r\mathbf{v} = \langle ra_1, \cdots, ra_n\rangle$.
  - Example: If $\mathbf{v} = \langle -1,2,2\rangle$ and $\mathbf{w} = \langle 3,0,-4\rangle$ then $2\mathbf{w} = \boxed{\langle 6,0,-8\rangle}$, and $\mathbf{v}+\mathbf{w} = \boxed{\langle 2,2,-2\rangle}$. Furthermore,$\mathbf{v}-2\mathbf{w} = \boxed{\langle -7,2,10\rangle}$.

- Arithmetic of vectors in $\mathbb{R}^n$ satisfies several algebraic properties that follow from the definition:
  - Addition of vectors is commutative ($\mathbf{v}+\mathbf{w} = \mathbf{w}+\mathbf{v}$) and associative ($\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$).
  - There is a zero vector $\mathbf{0}$ (namely, the vector with all entries zero), and every vector $\mathbf{v}$ has an additive inverse $-\mathbf{v}$ with $\mathbf{v}+(-\mathbf{v}) = \mathbf{0}$.
  - Scalar multiplication distributes over addition of vectors ($r(\mathbf{v}+\mathbf{w}) = r\mathbf{v}+r\mathbf{w}$) and scalars ($(r+s)\mathbf{v} = r\mathbf{v}+s\mathbf{v}$).

- With vectors in $\mathbb{R}^n$, we also have a quantity that resembles a product, called the dot product:

- Definition: The dot product of two vectors $\mathbf{v}_1 = \langle a_1, \ldots, a_n\rangle$ and $\mathbf{v}_2 = \langle b_1, \ldots, b_n\rangle$ is defined to be the scalar $\mathbf{v}_1 \cdot \mathbf{v}_2 = a_1b_1 + a_2b_2 + \cdots + a_nb_n$.
  - Note: The dot product of two vectors is a scalar, *not* a vector! (For this reason, the dot product is sometimes called the "scalar product" of two vectors.)
  - Example: The dot product $\langle 1,2\rangle \cdot \langle 3,4\rangle$ is $(1)(3)+(2)(4) = \boxed{11}$.
  - Example: The dot product $\langle -1,2,2\rangle \cdot \langle 3,0,-4\rangle$ is $(-1)(3)+(2)(0)+(2)(-4) = \boxed{-11}$.

- For any vectors $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$, and any scalar r, the following properties hold:
  - The dot product is commutative: $\mathbf{v}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{v}$.
  - The dot product distributes over addition and scaling: $(\mathbf{v}_1 + r\mathbf{v}_2)\cdot\mathbf{w} = (\mathbf{v}_1\cdot\mathbf{w}) + r(\mathbf{v}_2\cdot\mathbf{w})$.
  - The dot product is nonnegative: $\mathbf{v}\cdot\mathbf{v} \geq 0$ for any vector $\mathbf{v}$.

- One of the chief motivations of the dot product is to provide a way to measure angles between vectors. First, we must define the length of a vector:

- Definition: The norm (length, magnitude) of the vector $\mathbf{v} = \langle a_1, \ldots, a_n\rangle$ is $||\mathbf{v}|| = \sqrt{(a_1)^2 + \cdots + (a_n)^2}$.
  - This is an application of the distance formula: the norm of the vector $\langle a_1, \ldots, a_n\rangle$ is just the length of the line segment joining the origin $(0, \ldots, 0)$ to the point $(a_1, \ldots, a_n)$.
  - Example: For $\mathbf{v} = \langle -1,2,2\rangle$ and $\mathbf{w} = \langle 3,0,-4,5\rangle$, we have $||\mathbf{v}|| = \sqrt{(-1)^2+2^2+2^2} = \boxed{3}$, and $||\mathbf{w}|| = \sqrt{3^2+0^2+(-4)^2+5^2} = \boxed{\sqrt{50}}$.

- If r is a scalar, we can see immediately from the definition that $||r\,\mathbf{v}|| = |r|\;||\mathbf{v}||$, since we can just factor $\sqrt{r^2} = |r|$ from each term under the square root.
- Observe also that the dot product of a vector with itself is the square of the norm: $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$.

- Using the dot product, we can give a formula for the angle between two vectors:
- Theorem (Dot Product): For vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ forming an angle θ between them, $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1||\;||\mathbf{v}_2|| \cos(\theta)$.
  - Proof: Consider the triangle formed by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_2 - \mathbf{v}_1$: applying the Law of Cosines in this triangle yields $||\mathbf{v}_2 - \mathbf{v}_1||^2 = ||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 - 2\,||\mathbf{v}_1||\,||\mathbf{v}_2|| \cos(\theta)$.
  - Since the square of the norm is the dot product of a vector with itself, and the dot product is distributive, $||\mathbf{v}_2 - \mathbf{v}_1||^2 = (\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1) = (\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{v}_1 \cdot \mathbf{v}_2) - (\mathbf{v}_2 \cdot \mathbf{v}_1) + (\mathbf{v}_1 \cdot \mathbf{v}_1) = ||\mathbf{v}_2||^2 - 2(\mathbf{v}_1 \cdot \mathbf{v}_2) + ||\mathbf{v}_1||^2$.
  - Now by comparing to the Law of Cosines expression, we must have $||\mathbf{v}_1||\,||\mathbf{v}_2|| \cos(\theta) = \mathbf{v}_1 \cdot \mathbf{v}_2$, as claimed.

- Using the Dot Product Theorem, we can compute the angle between two vectors.
- Example: Compute the angle between the vectors $\mathbf{v} = \langle 2\sqrt{2}, 1, \sqrt{3} \rangle$ and $\mathbf{w} = \langle 0, \sqrt{3}, 1 \rangle$.
  - We compute $\mathbf{v} \cdot \mathbf{w} = (2\sqrt{2})(0) + (1)(\sqrt{3}) + (\sqrt{3})(1) = 2\sqrt{3}$, and $||\mathbf{v}|| = \sqrt{(2\sqrt{2})^2 + 1^2 + (\sqrt{3})^2} = 2\sqrt{3}$ and $||\mathbf{w}|| = \sqrt{(\sqrt{3})^2 + 0^2 + 1^2} = 2$.
  - Then by the Dot Product Theorem, the angle θ between the vectors satisfies $2\sqrt{3} = 2 \cdot 2\sqrt{3} \cdot \cos(\theta)$, meaning that $\theta = \cos^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{3}}$.

0.3 Complex Numbers, Fields

- Definitions: A complex number is a number of the form $a + bi$, where a and b are real numbers and i is the "imaginary unit", defined so that $i^2 = -1$. The set of all complex numbers is denoted $\mathbb{C}$ ("complex").
  - Notation: Sometimes, i is written as $\sqrt{-1}$. In certain disciplines (especially electrical engineering), the letter j can be used to denote $\sqrt{-1}$, rather than i (which is used to denote electrical current).
  - The real part of $z = a + bi$, denoted $\mathrm{Re}(z)$, is the real number a.
  - The imaginary part of $z = a + bi$, denoted $\mathrm{Im}(z)$, is the real number b.
  - The complex conjugate of $z = a + bi$, denoted[5] as $\overline{z}$, is the complex number $a - bi$.
  - The modulus (also called the absolute value, magnitude, or length) of $z = a + bi$, denoted $|z|$, is the real number $\sqrt{a^2 + b^2}$. Observe that $|z|^2 = a^2 + b^2 = z\overline{z}$.
  - Example: $\mathrm{Re}(4 - 3i) = 4$, $\mathrm{Im}(4 - 3i) = -3$, $\overline{4 - 3i} = 4 + 3i$, $|4 - 3i| = 5$.

- Two complex numbers are added (or subtracted) simply by adding (or subtracting) their real and imaginary parts: $(a + bi) + (c + di) = (a + c) + (b + d)i$.
  - Example: The sum of $1 + 2i$ and $3 - 4i$ is $\boxed{4 - 2i}$. The difference is $(1 + 2i) - (3 - 4i) = \boxed{-2 + 6i}$.

- Two complex numbers are multiplied using the distributive law and the fact that $i^2 = -1$: $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$.
  - Example: The product of $1 + 2i$ and $3 - 4i$ is $(1 + 2i)(3 - 4i) = 3 + 6i - 4i - 8i^2 = \boxed{11 + 2i}$.

- For division, we rationalize the denominator: $\dfrac{a + bi}{c + di} = \dfrac{(a + bi)(c - di)}{(c + di)(c - di)} = \dfrac{ac + bd}{c^2 + d^2} + \dfrac{bc - ad}{c^2 + d^2}\,i$.

[5] The notation for the complex conjugate varies among disciplines. In physics and computer science, the notation z^* often denotes the complex conjugate (among other reasons, because it is easier to typeset).

- Example: The quotient of $2i$ by $1-i$ is $\dfrac{2i}{1-i} = \dfrac{2i(1+i)}{(1-i)(1+i)} = \dfrac{-2+2i}{2} = \boxed{-1+i}$.

- A key property of the conjugate is that it is multiplicative: if $z = a+bi$ and $w = c+di$, then $\overline{zw} = \overline{z} \cdot \overline{w}$. (This is easy to see just by multiplying out the relevant quantities.) From this we see that the modulus is also multiplicative: $|zw| = |z| \cdot |w|$.

  - Example: If $z = 1+2i$ and $w = 3-i$, then $\overline{z} = 1-2i$ and $\overline{w} = 3+i$. We compute $zw = 5+5i$ and $\overline{z} \cdot \overline{w} = 5-5i$, so indeed $\overline{zw} = \overline{z} \cdot \overline{w}$. Furthermore, we have $|z| = \sqrt{5}$,$|w| = \sqrt{10}$, and $|zw| = \sqrt{50} = |z| \cdot |w|$.
  - This is the underlying reason for why division works in general: we write $\dfrac{z}{w} = \dfrac{z \cdot \overline{w}}{w \cdot \overline{w}} = \dfrac{z \cdot \overline{w}}{|w|^2}$, where the denominator is now the real number $|w|^2 = c^2 + d^2$.

- For reference, here are a few more properties of complex numbers (each of which can be verified by writing everything out in terms of real and imaginary parts):

  - For any z and w, $\overline{z+w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z} \cdot \overline{w}$, and $\overline{\overline{z}} = z$.
  - For any z, $z = \overline{z}$ if and only if z is real, and $\overline{z} = -\overline{z}$ if and only if z is purely imaginary (of the form ri where r is real).
  - For any z and w, $\mathrm{Re}(z) \leq |z|$, $\mathrm{Im}(z) \leq |z|$, $|zw| = |z| \cdot |w|$, and $|z+w| \leq |z| + |w|$. (The latter follows by observing $|z+w|^2 = (z+w)(\overline{z+w}) = |z|^2 + |w|^2 + 2\mathrm{Re}(z\overline{w}) \leq |z|^2 + |w|^2 + 2|zw| = (|z| + |w|)^2$.)

- The real numbers and the complex numbers are both examples of fields: sets of numbers that can be added, subtracted, multiplied, and divided (except by zero) and possess various algebraic relations involving these operations. More formally:

- Definition: A field is an ordered triple $(F, +, \cdot)$ consisting of a set of numbers F together with two binary operations[6], $+$ (addition) and $\cdot$ (multiplication), satisfying the following axioms:

  [F1] Addition is associative: $a + (b + c) = (a + b) + c$ for any elements a, b, c in F.

  [F2] Addition $+$ is commutative: $a + b = b + a$ for any elements a, b in F.

  [F3] There is an additive identity 0 satisfying $a + 0 = a$ for all a in F.

  [F4] Every element a in F has an additive inverse $-a$ satisfying $a + (-a) = 0$.

  [F5] Multiplication is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for any elements a, b, c in F.

  [F6] Multiplication is commutative: $a \cdot b = b \cdot a$ for any elements a, b in F.

  [F7] There is a multiplicative identity $1 \neq 0$, satisfying $1 \cdot a = a = a \cdot 1$ for all a in F.

  [F8] Every nonzero a in F has a multiplicative inverse a^{-1} satisfying $a \cdot a^{-1} = 1$.

  [F9] Multiplication distributes over addition: $a \cdot (b + c) = a \cdot b + a \cdot c$ for any elements a, b, c in F.

- Subtraction and division are not explicitly part of the axioms of a field. They are instead obtained in terms of addition and multiplication, per the axioms.

  - Explicitly, in a field we define $a - b = a + (-b)$ and $a/b = a \cdot (b^{-1})$, the latter whenever $b \neq 0$.

- It will not be necessary for us to work with this "axiomatic" description of a field: we will only need to use these properties in the context of vector spaces.

- Here are some examples of fields:

  - Example: The sets of rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, and complex numbers $\mathbb{C}$ are fields.
  - Example: The set $\mathbb{Q}(\sqrt{2})$ of real numbers of the form $x + y\sqrt{2}$, where x and y are rational numbers, is a field.

[6] The result of applying these operations to elements a and b is denoted by $a + b$ and $a \cdot b$ (or simply ab), respectively. The definition of "binary operation" means that $a + b$ and $a \cdot b$ are also numbers in F.

- Example: If p is a prime number, the set $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \{\overline{0}, \overline{1}, \ldots, \overline{p-1}\}$ of integers modulo p is a field. Explicitly, $\mathbb{Z}/p\mathbb{Z}$ is the set of residue classes of integers modulo p, in which we identify any two integers that differ by a multiple of p, with the usual addition and multiplication operations inherited from $\mathbb{Z}$. For example, if $p = 5$, we have $\overline{2} + \overline{3} = \overline{0}$ (because $2 + 3 = 5$, and we consider 5 to be the same as 0 modulo 5) and $\overline{2} \cdot \overline{4} = \overline{3}$. Unlike the other examples above, these fields have only finitely many elements.

- There is another fundamental quantity attached to a field known as its characteristic:

- Definition: If F is a field, we say F has characteristic p if $p1_F = 0$, and no smaller positive integer multiple of 1 is 0. (Recall that $p1_F = \underbrace{1_F + 1_F + \cdots + 1_F}_{p \text{ times}}$.) If $n1_F \neq 0$ for all $n > 0$, then we say F has characteristic 0.
  - Example: For a prime p, the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ has characteristic p, while the fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ have characteristic 0.
  - Any finite field necessarily has positive characteristic, although infinite fields with positive characteristic also exist.

- Proposition (Positive Characteristic): If the field F has characteristic $p > 0$, then p is a prime.
  - Proof: Suppose F has characteristic $m > 0$ and $m = ab$ for positive integers a, b: then $0 = m1_F = (a1_F) \cdot (b1_F)$.
  - Since F is a field, this implies that one of $a1_F$ and $b1_F$ must be zero, but since m is minimal, the only possibility is that $a = m$ or $b = m$, meaning that m must be prime.

0.4 Matrices, Systems of Linear Equations, and Determinants

- Let F be a field.

- Definition: An $m \times n$ matrix is an array of numbers (from F) with m rows and n columns. A square matrix is one with the same number of rows and columns: that is, an $n \times n$ matrix for some n. The set of all $m \times n$ matrices with elements in F is denoted $M_{m \times n}(F)$.
  - Examples: $\begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \end{bmatrix}$ is a 2×3 real matrix, and $\begin{bmatrix} \pi & 0 & 0 \\ 0 & 4i & 0 \\ 0 & 0 & 6 \end{bmatrix}$ is a 3×3 complex matrix.

- Definition: If A is a matrix, the entry in the ith row and jth column of A is called the (i,j)-entry of A, and will be denoted $a_{i,j}$.
  - Warning: It is easy to mix up the coordinates. Remember that the first coordinate specifies the row, and the second coordinate specifies the column.
  - Example: If $A = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$, then the $(2,2)$-entry is $a_{2,2} = 0$ and the $(1,3)$-entry is $a_{1,3} = 4$.

- Definition: In an $n \times n$ square matrix, the (i,i)-entries for $1 \leq i \leq n$ form the diagonal of the matrix. The trace of a square matrix is the sum of its diagonal entries.
  - Example: The diagonal entries of $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ are 4 and 2, so the trace of this matrix is 6.

0.4.1 Matrix Arithmetic

- Like with vectors, we can add and subtract matrices of the same size, and we can also multiply a matrix by a scalar. Each of these operations is done "componentwise": to add or subtract, we just add or subtract the corresponding entries of the two matrices. To multiply by a scalar, we just multiply each entry by that scalar.

- Example: If $A = \begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, then $A + B = \begin{bmatrix} 1+3 & 6+0 \\ 2+0 & 2+2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}$, $2A = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 6 \\ 2 \cdot 2 & 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 4 & 4 \end{bmatrix}$, and $A - \frac{1}{3}B = \begin{bmatrix} 0 & 6 \\ 2 & \frac{4}{3} \end{bmatrix}$.

- We also have a transposition operation, where we interchange the rows and columns of the matrix:

- Definition (Matrix Transpose): If A is an $n \times m$ matrix, then the transpose of A, denoted A^T, is the $m \times n$ matrix whose (i, j)-entry is equal to the (j, i)-entry of A.

  - Example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.
  - A symmetric matrix is one with $A = A^T$, and a skew-symmetric matrix is one with $A = -A^T$. (For either of these properties to hold, A must be a square matrix.)

- We can also multiply matrices in certain situations. However, matrix multiplication, unlike matrix addition, is NOT performed componentwise.

  - The definition of matrix multiplication seems strange and arbitrary at first. However, there is a very good reason for this peculiar definition, and we will explain it when we discuss linear transformations.

- Definition (Matrix Product): If A is an $m \times n$ matrix and B is an $n \times q$ matrix, then the matrix product $A \cdot B$, often written simply as AB, is the $m \times q$ matrix whose (i, j)-entry is the sum $(AB)_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$, the sum of products of corresponding entries from the ith row of A with the jth column of B.

  - Important Note: In order for the matrix product to exist, the number of columns of A must equal the number of rows of B. In particular, if A and B are the same size, their product exists only if they are square matrices. Also, if AB exists, then BA may not necessarily exist.
  - More compactly, the (i, j) entry of the matrix product AB is the dot product of the ith row of A with the jth column of B (each thought of as vectors). These two vectors must have the same length for the dot product to exist, which is why the length of each row of A (the number of columns of A) must equal the length of each column of B (the number of rows of B).
  - This product is sometimes called the row-column product to emphasize the fact that it is a product involving the rows of A with the columns of B.

- Example: If $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 3 \end{bmatrix}$, find AB and BA.

  - Since A is a 2×3 matrix and B is a 3×2 matrix, we see that AB is defined and will be a 2×2 matrix.
  - The $(1, 1)$ entry of AB is $(-1)(1) + (1)(2) + (2)(3) = 7$.
  - The $(1, 2)$ entry of AB is $(-1)(2) + (1)(0) + (2)(3) = 4$.
  - The $(2, 1)$ entry of AB is $(0)(1) + (1)(2) + (1)(3) = 5$.
  - The $(2, 2)$ entry of AB is $(0)(2) + (1)(0) + (1)(3) = 3$.
  - Putting all of this together gives $AB = \boxed{\begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}}$.
  - We see that $BA = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is also defined and will be a 3×3 matrix.
  - The $(1, 1)$ entry of BA is $(1)(-1) + (2)(0) = -1$, the $(1, 2)$ entry is $(1)(1) + (2)(1) = 3$, and the $(1, 3)$ entry is $(1)(2) + (2)(1) = 4$.

- In a similar way we can compute the other six entries: the result is $BA = \boxed{\begin{bmatrix} -1 & 3 & 4 \\ -2 & 2 & 4 \\ -3 & 6 & 9 \end{bmatrix}}$.

- If we restrict our attention to square matrices, then matrices under addition and multiplication obey some, but not all, of the algebraic properties that real numbers do.
  - In general, matrix multiplication is NOT commutative: AB typically isn't equal to BA, even if A and B are both square matrices.
    - Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. Then $AB = \begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix}$ while $BA = \begin{bmatrix} 11 & 16 \\ 19 & 28 \end{bmatrix}$.
  - Matrix multiplication distributes over addition, on both sides: $(A+B)C = AC + BC$ and $A(B+C) = AB + AC$.
    - This property can be derived from the definition of matrix multiplication, along with some arithmetic.
  - Matrix multiplication is associative: $(AB)C = A(BC)$, if A, B, C are of the proper dimensions.
    - In particular, taking the nth power of a square matrix is well-defined for every positive integer n.
    - This property can also be derived from the definition, but the arithmetic is very cumbersome.
  - The transpose of the product of two matrices is the product of their transposes in reverse order: $(AB)^T = B^T A^T$.
    - This property can likewise be derived from the definition of matrix multiplication.
  - The trace distributes over addition: $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$. Also, $\mathrm{tr}(cA) = c\,\mathrm{tr}(A)$ for any c.

- Definition: If A is an $n \times n$ matrix, then there is a zero matrix Z_n which has the properties $Z_n + A = A$ and $Z_n A = A Z_n = Z_n$. This matrix Z_n is the matrix whose entries are all zeroes.
  - Example: The 2×2 zero matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
  - Remark: In contrast to real (or complex) numbers, where $x^2 = 0$ implies $x = 0$, there exist nonzero matrices whose square is nonetheless the zero matrix. One such matrix is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$: it is easy to check that A^2 is the zero matrix, but of course A itself is nonzero.

- Definition: If A is an $n \times n$ matrix, then there is an $n \times n$ identity matrix I_n which has the property that $I_n A = A I_n = A$. This matrix I_n is the matrix whose diagonal entries are 1s and whose other entries are 0s.
  - Example: The 2×2 identity matrix is $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the 3×3 identity matrix is $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
  - Observe that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

0.4.2 Systems of Linear Equations

- We can use matrices to solve systems of linear equations, like the system $\left\{ \begin{array}{rcrcr} x & + & y & = & 7 \\ 2x & - & 2y & = & -2 \end{array} \right\}$. (The braces are only there to emphasize that the equations are to be considered together.)
  - Recall that a linear equation is an equation of the form $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$, for some constants $a_1, \ldots a_n, b$, and variables $x_1, \ldots, x_n$.
  - When we seek to find the solutions to a system of linear equations, this means finding all possible values of the variables such that all equations are satisfied simultaneously.

- The traditional method for solving a system of linear equations (likely familiar from basic algebra) is by elimination: we solve the first equation for one variable (say, x) in terms of the others, and then plug in the result to all the other equations to obtain a reduced system involving one fewer variable. Eventually, the system will simplify either to a contradiction (e.g., $1 = 0$), a unique solution, or an infinite family of solutions.

- Example: Solve the system $\left\{ \begin{array}{rcrcr} x & + & y & = & 7 \\ 2x & - & 2y & = & -2 \end{array} \right\}$.
  - We can solve the first equation for x to obtain $x = 7 - y$.
  - Plugging in this relation to the second equation gives $2(7 - y) - 2y = -2$, or $14 - 4y = -2$, so that $\boxed{y = 4}$. Then since $x = 7 - y$ we obtain $\boxed{x = 3}$.

- Another way to perform elimination is to add and subtract multiples of the equations, so to eliminate variables (and remove the need to solve for each individual variable before eliminating it).
  - In the example above, instead of solving the first equation for x, we could multiply the first equation by -2 and then add it to the second equation, so as to eliminate x from the second equation.
  - This yields the same overall result, but is less computationally difficult.
  - This procedure of elimination can be simplified even more, because we don't really need to write the variable labels down every time. We only need to keep track of the coefficients, which we can do by putting them into a "coefficient matrix".
  - For example, $\left\{ \begin{array}{rcrcrcr} x & + & y & + & 3z & = & 4 \\ 2x & + & 3y & - & z & = & 1 \\ -x & + & 2y & + & 2z & = & 1 \end{array} \right\}$ has associated coefficient matrix $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 3 & -1 & 1 \\ -1 & 2 & 2 & 1 \end{array} \right]$.
  - When working with a coefficient matrix, we will draw a line to separate the coefficients of the variables from the constant terms. This type of matrix is often called an augmented matrix.

- When doing elimination, each step involves one of the three elementary row operations on the rows of the coefficient matrix:
  1. Interchange two rows.
  2. Multiply all entries in a row by a nonzero constant.
  3. Add a constant multiple of one row to another row.

- Each of these elementary row operations leaves unchanged the solutions to the associated system of linear equations. The idea of elimination is to apply these elementary row operations to the coefficient matrix until it is in a simple enough form that we can simply read off the solutions to the original system of equations.
  - When we use elementary row operations on a matrix, we will indicate the type of operation along with an arrow.

- Example: Solve the system $\left\{ \begin{array}{rcrcrcr} x & + & y & + & 3z & = & 4 \\ 2x & + & 3y & - & z & = & 1 \\ -x & + & 2y & + & 2z & = & 1 \end{array} \right\}$ using elimination.
  - The associated coefficient matrix for this system is $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 3 & -1 & 1 \\ -1 & 2 & 2 & 1 \end{array} \right]$.
  - We apply elementary row operations to clear out the first column:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 3 & -1 & 1 \\ -1 & 2 & 2 & 1 \end{array} \right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & -7 & -7 \\ -1 & 2 & 2 & 1 \end{array} \right] \xrightarrow{R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & -7 & -7 \\ 0 & 3 & 5 & 5 \end{array} \right].$$

  - Now we are done with the first column and can focus on the other columns:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & -7 & -7 \\ 0 & 3 & 5 & 5 \end{array} \right] \xrightarrow{R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & -7 & -7 \\ 0 & 0 & 26 & 26 \end{array} \right] \xrightarrow{\frac{1}{26} R_3} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & -7 & -7 \\ 0 & 0 & 1 & 1 \end{array} \right].$$

- We can now solve the system from the bottom up: the bottom row yields $z = 1$. The middle row then says $y - 7z = -7$ from which we see $y = 0$. Finally, the top row says $x + y + 3z = 4$, so that $x = 1$.
- Thus, the system has the unique solution $\boxed{x = 1,\ y = 0,\ z = 1}$.

- We can further simplify our approach to systems of linear equations by rewriting a system of linear equations as a matrix equation (using matrix multiplication).
  - For example, $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & -1 \\ -1 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y + 3z \\ 2x + 3y - z \\ -x + 2y + 2z \end{bmatrix}$, so the system $\left\{ \begin{array}{rcrcrcl} x & + & y & + & 3z & = & 4 \\ 2x & + & 3y & - & z & = & 1 \\ -x & + & 2y & + & 2z & = & 1 \end{array} \right\}$ is equivalent to the single matrix equation $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & -1 \\ -1 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$.

0.4.3 Elementary Matrices and Echelon Forms

- Each of the elementary row operations on an $n \times n$ matrix corresponds to left-multiplication by the matrix obtained by applying the corresponding row operation to the identity matrix.

- Definition: An elementary row matrix is a matrix obtained by performing a single elementary row operation on the identity matrix.
  - Examples: The matrices $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$ are elementary matrices associated to the operations "add twice row 2 to row 1", "double row 1", "swap rows 2 and 3", and "add 3 times row 2 to row 3", respectively.

- The procedure of applying elementary row operations to a matrix until it is in a simpler form is called row-reduction. We can give a more precise definition of what "simpler" means:

- Definition: A matrix is in row-echelon form if (i) all rows with at least one nonzero element are above any row of all zero entries, and (ii) the first nonzero term in each row is always to the right of the first nonzero term in the row above it. (The first nonzero term in each row is called the pivot.)
  - A shorter way of writing the two conditions is (i) all rows without a pivot (the rows of all zeroes) are at the bottom, and (ii) any row's pivot, if it has one, lies to the right of the pivot of the row directly above it.

- Here are some examples of matrices in row-echelon form, where the pivot elements have been boxed:
  - $\begin{bmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & \boxed{1} & 0 & 1 \end{bmatrix}, \begin{bmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}, \begin{bmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \boxed{3} & 4 & 5 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

- Here are some examples of matrices not in row-echelon form:
  - $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$: the pivot in the second row is not strictly to the right of the pivot element above it.
  - $\begin{bmatrix} 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$: the pivot in the third row is not strictly to the right of the pivot element above it.
  - $\begin{bmatrix} 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$: the row of all zeroes is not at the bottom of the matrix.

- If the coefficient matrix is in row-echelon form, it is easy to read off the solutions to the corresponding system of linear equations by working from the bottom up.
  - Example: The augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right]$ corresponding to the system

$$\begin{array}{rcrcrcl} x & + & y & + & 3z & = & 4 \\ & & y & - & z & = & 1 \\ & & & & 2z & = & 4 \end{array}.$$

    is in row-echelon form. The bottom equation immediately gives $z = 2$. Then the middle equation gives $y = 1 + z = 3$, and the top equation gives $x = 4 - y - 3z = -5$.

- Example: Use row operations to put the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}$ into row-echelon form.
  - We apply elementary row operations to clear out the first column, and then we can clear out the third row.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_4 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

  - The matrix is now in row-echelon form.
  - Notice that there are other possible combinations of row operations we could have performed to put the matrix into a row-echelon form.
  - For example, we could have started by swapping rows 1 and 3, and then cleared out the first column and the lower rows:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_4 - 2R_1} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -4 & -6 \end{bmatrix} \xrightarrow{R_4 + 2R_2} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

  - Notice that we obtain a different row-echelon form in this case.

- An even simpler form is called reduced row-echelon form:
- Definition: A matrix is in reduced row-echelon form if it is in row-echelon form, all pivot elements are equal to 1, and each pivot element is the only nonzero term in its column.
  - Here are some matrices in reduced row-echelon form: $\begin{bmatrix} \boxed{1} & 0 & 0 & 4 & 5 \\ 0 & \boxed{1} & 0 & 3 & 4 \\ 0 & 0 & \boxed{1} & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \boxed{1} & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} \boxed{1} & 2 & 0 & 4 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$.

- Example: Use row operations to put the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \\ 5 & 2 & 0 \end{bmatrix}$ into reduced row-echelon form.

- Applying elementary row operations to clear out the first column, and then the bottom row, yields

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \\ 5 & 2 & 0 \end{bmatrix} \xrightarrow{R_2-3R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 5 & 2 & 0 \end{bmatrix} \xrightarrow{R_3-5R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 2 & -10 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}.$$

- There are numerous ways to row-reduce a given matrix until it is in row-echelon form, and many different row-echelon forms are possible. However, it turns out that the reduced row-echelon form is always unique:
- Theorem: Every matrix has a unique reduced row-echelon form.
  - We will not prove this theorem (the proof is not difficult, but requires developing some results on the solution spaces of systems of linear equations).
  - However, it is useful from a theoretical standpoint to know that, regardless of the way we perform row-reductions, we will always obtain the same reduced row-echelon form when we are finished.
- Definition: The rank of a matrix is the number of nonzero rows in its (reduced) row-echelon form. Equivalently, it is the number of pivots that appear when the matrix is in (reduced) row-echelon form.
  - Examples: The rank of $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$ is 3, while the rank of $\begin{bmatrix} 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is 2.
- By using row operations on a coefficient matrix, we can find the solutions to the associated system of equations, since as we noted before each of the elementary row operations does not change the solutions to the system.

0.4.4 The Inverse of a Matrix

- Given a square $n \times n$ matrix A, we might like to know whether it has a multiplicative inverse.
- Definition: If A is an $n \times n$ square matrix, then we say A is invertible (or nonsingular) if there exists an $n \times n$ matrix A^{-1}, the inverse matrix, such that $AA^{-1} = A^{-1}A = I_n$, where I_n is the $n \times n$ identity matrix. If no such matrix A^{-1} exists, we say A is not invertible (or singular).
  - Example: The matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ has inverse matrix $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$, since we can compute $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.
  - Non-Example: The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has no multiplicative inverse, since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ times any matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and cannot be the identity matrix.
  - Non-Example: The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has no multiplicative inverse, since $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$, which is never equal to the identity matrix for any choice of a, b, c, d since the top and bottom rows are always equal.
  - Example: Elementary row matrices are invertible, with the inverse being the elementary row matrix for the inverse row operation.
- Here are a few basic properties of inverse matrices:
- Proposition (Properties of Inverses): If A and B are invertible $n \times n$ matrices, then the following hold:

1. If a matrix is invertible, then it has only one inverse matrix.
   - Proof: Suppose B_1 and B_2 both had $AB_1 = I_n = B_1A$ and $AB_2 = I_n = B_2A$.
   - Then $B_1 = B_1I_n = B_1(AB_2) = (B_1A)B_2 = I_nB_2 = B_2$ and so $B_1 = B_2$.
2. If A is invertible, then so is A^{-1}, and $(A^{-1})^{-1} = A$.

- Proof: Since $A^{-1}A = I_n = AA^{-1}$ we see that A also fulfills the role of the inverse of A^{-1}.

3. If A and B are invertible, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$. More generally, $(A_1A_2\cdots A_n)^{-1} = A_n^{-1}\cdots A_2^{-1}A_1^{-1}$.
   - Proof: We simply compute $(AB)(B^{-1}A^{-1}) = A(B\,B^{-1})A^{-1} = A(I_n)A^{-1} = A\,A^{-1} = I_n$. Similarly, the product in the other order will also come out to be the identity matrix.
   - The second result follows from the first one by an easy induction.
4. The 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, and if so, the inverse is given by $\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
   - Proof: This follows simply from solving the system of equations for e, f, g, h in terms of a, b, c, d that arises from comparing entries in the product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$: one obtains precisely the solution given above. If $ad = bc$ then the system is inconsistent and there is no solution; otherwise there is exactly one solution as given.
5. Any square matrix with a row or column of all zeroes cannot be invertible.
   - Proof: Suppose the $n \times n$ matrix C has all entries in its ith row equal to zero. Then for any $n \times n$ matrix D, the product CD will have all entries in its ith row equal to zero, so it cannot be the identity matrix.
   - Similarly, if the $n \times n$ matrix C has all entries in its ith column equal to zero, then for any $n \times n$ matrix D, the product DC will have all entries in its ith column equal to zero.
6. The matrix A is invertible if and only if it is row-equivalent to the identity matrix I_n.
   - Proof: Consider the reduced row-echelon form of the matrix A. Because A is a square matrix, the reduced row-echelon form is either the identity matrix, or a matrix with a row of all zeroes.
   - Suppose A is row-equivalent to the identity matrix. Each elementary row operation corresponds to left-multiplication by an invertible matrix, so there are elementary matrices E_i with $1 \leq i \leq k$ such that $E_kE_{k-1}\cdots E_1A = I_n$.
   - So if we let $B = E_kE_{k-1}\cdots E_1$, then B is invertible (its inverse is $B^{-1} = E_1^{-1}\cdots E_{k-1}^{-1}E_k^{-1}$) and $BA = I_n$.
   - Multiplying the expression $BA = I_n$ on the left by B^{-1} and on the right by B produces $AB = B^{-1}B = I_n$, so we see $AB = BA = I_n$. Thus B is the inverse of A, as claimed.
   - Now suppose that A is not row-equivalent to the identity matrix. Then its reduced row-echelon form A_{red} must contain a row of all zero entries. From our results above we see that A_{red} cannot be invertible, and since $A = E_1E_2\ldots E_kA_{red}$, then if A had an inverse B then A_{red} would have an inverse, namely $BE_1E_2\cdots E_k$.

- From the proof of this theorem we see that if A has an inverse, we can compute it as the composition of the appropriate row operations that convert A into the identity matrix. Explicitly, in order to compute the inverse of an $n \times n$ matrix A using row reduction (or to see if it is non-invertible), we can perform the following procedure, called Gauss-Jordan elimination:
  - Step 1: Set up a "double" matrix $[A\,|I_n]$ where I_n is the identity matrix.
  - Step 2: Perform row operations to put A in reduced row-echelon form. (Carry the computations through on the entire matrix, but only pay attention to the left side when deciding what operations to do.)
  - Step 3: If A can be row-reduced to the $n \times n$ identity matrix, then row-reducing A will produce the double matrix $[I_n\,|A^{-1}]$. If A cannot be row-reduced to the $n \times n$ identity matrix, then A is not invertible.

- Example: Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & 2 & -5 \end{bmatrix}$.
  - First, we set up the starting matrix $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -5 & 0 & 0 & 1 \end{array}\right]$.

- Now we perform row operations to row-reduce the matrix on the left:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & 2 & -5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3+2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 2 & 1 \end{array}\right]$$

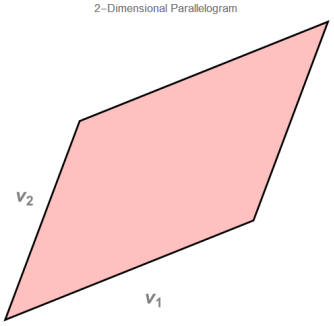
$$\begin{array}{c} R_2-3R_3 \\ \longrightarrow \\ R_1+R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 1 \\ 0 & -1 & 0 & 10 & -5 & -3 \\ 0 & 0 & 1 & -4 & 2 & 1 \end{array}\right] \xrightarrow{(-1)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 1 \\ 0 & 1 & 0 & -10 & 5 & 3 \\ 0 & 0 & 1 & -4 & 2 & 1 \end{array}\right].$$

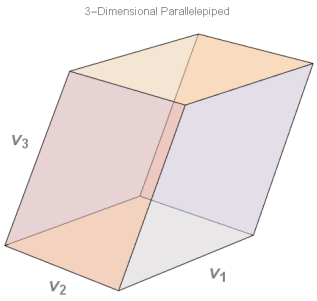
- We have row-reduced A to the identity matrix, so A is invertible and $\boxed{A^{-1} = \left[\begin{array}{ccc} -3 & 2 & 1 \\ -10 & 5 & 3 \\ -4 & 2 & 1 \end{array}\right]}$.

0.4.5 The Determinant of a Matrix

- We might like to know, without performing all of the row-reductions, if a given large matrix is invertible. This motivates the idea of the determinant, which will tell us precisely when a matrix is invertible.
- Definition: The determinant of a square matrix A, denoted $\det(A)$ or $|A|$, is defined inductively. For a 1×1 matrix $[a]$ it is just the constant a. For an $n \times n$ matrix we compute the determinant via "cofactor expansion": define $A^{(1,k)}$ to be the matrix obtained from A by deleting the 1st row and kth column. Then $\det(A) = \sum_{k=1}^{n} (-1)^{k+1} a_{1,k} \det(A^{(1,k)})$.
  - The best way to understand determinants is to work out some examples.
- Example: The determinant $\left|\begin{array}{cc} a & b \\ c & d \end{array}\right|$ is given by $\left|\begin{array}{cc} a & b \\ c & d \end{array}\right| = ad - bc$.
  - So, as particular cases, $\left|\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right| = (1)(4) - (2)(3) = -2$ and $\left|\begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array}\right| = (1)(2) - (1)(2) = \boxed{0}$.
- Example: The determinant $\left|\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right|$ is given by $\left|\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right| = a_1 \left|\begin{array}{cc} b_2 & b_3 \\ c_2 & c_3 \end{array}\right| - a_2 \left|\begin{array}{cc} b_1 & b_3 \\ c_1 & c_3 \end{array}\right| + a_3 \left|\begin{array}{cc} b_1 & b_2 \\ c_1 & c_2 \end{array}\right|$.
  - As a particular case, $\left|\begin{array}{ccc} 1 & 2 & 4 \\ -1 & 1 & 0 \\ -2 & 1 & 3 \end{array}\right| = 1\left|\begin{array}{cc} 1 & 0 \\ 1 & 3 \end{array}\right| - 2\left|\begin{array}{cc} -1 & 0 \\ -2 & 3 \end{array}\right| + 4\left|\begin{array}{cc} -1 & 1 \\ -2 & 1 \end{array}\right| = 1(3) - 2(-3) + 4(1) = \boxed{13}$.
- There is a nice way to interpret the determinant geometrically:
- Proposition: If $\mathbf{v}_1$,$\mathbf{v}_2$ are vectors in $\mathbb{R}^2$, then the determinant of the matrix whose rows are $\mathbf{v}_1, \mathbf{v}_2$ is the signed area of the parallelogram formed by $\mathbf{v}_1$,$\mathbf{v}_2$. Furthermore, if $\mathbf{w}_1$,$\mathbf{w}_2$,$\mathbf{w}_3$ are vectors in $\mathbb{R}^3$, then the determinant of the matrix whose rows are $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ is the signed volume of the parallelepiped (skew box) formed by $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.
  - Note that a signed area (and a signed volume) can be negative: the sign indicates the relative orientation of the vectors. For two vectors, the signed area is positive if the second vector is counterclockwise from the first one and negative otherwise. For three vectors, the signed volume is positive if the vectors are arranged per the right-hand rule: align the fingers of the right hand along the first vector and then curl them into the direction of the second vector; the orientation is positive if the thumb is pointing in the direction of the third vector, and negative if it is pointing in the opposite direction.

- Here are pictures of the appropriate regions:





- The proof is a straightforward geometric calculation, which we omit.

- The determinant behaves in a very predictable way under the elementary row operations (showing these results requires a more careful technical analysis of the determinant and so we will omit the details):
  - Interchanging two rows multiplies the determinant by -1.
    - Example: $\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} = 5$ while $\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} = -5$.
  - Multiplying all entries in one row by a constant scales the determinant by the same constant.
    - Example: $\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} = 5$ while $\begin{vmatrix} 9 & 6 \\ -1 & 1 \end{vmatrix} = 3 \cdot 5 = 15$.
  - Adding or subtracting a scalar multiple of one row to another leaves the determinant unchanged.
    - Example: $\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} = 5$ while $\begin{vmatrix} 3+2(-1) & 2+2(1) \\ -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ -1 & 1 \end{vmatrix} = 5$.

- From the above analysis we can deduce a number of other properties of determinants (these are also straightforward algebraic arguments, which we omit):

- Proposition (Properties of Determinants): If A and B are $n \times n$ matrices, the following properties hold:

1. If A has a row or column of all zeroes, then $\det(A) = 0$.
   - Example: $\begin{vmatrix} 3 & 2 \\ 0 & 0 \end{vmatrix} = 0$ and $\begin{vmatrix} 3 & 0 \\ -1 & 0 \end{vmatrix} = 0$.
2. If one row or column of A is a scalar multiple of another, then $\det(A) = 0$. More generally, if the matrix is row-equivalent to a matrix with a row or column of all zeroes, then the determinant is zero.
   - Example: $\begin{vmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 0$ (rows 1,2) and $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 1 & 1 & 3 \end{vmatrix} = 0$ (columns 1,3).
3. The determinant is multiplicative: $\det(AB) = \det(A)\det(B)$.
   - Example: If $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$, then $AB = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$. We see $\det(A) = 2$, $\det(B) = 3$, and $\det(AB) = 6$.
   - Example: If $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, then $AB = \begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix}$. We see $\det(A) = 5$, $\det(B) = 10$, and $\det(AB) = 50$.
4. The determinant of any upper-triangular matrix (a matrix whose entries below the diagonal are all zeroes) is equal to the product of the diagonal entries.
   - Example: $\begin{vmatrix} 6 & -1 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 36$.
5. The matrix A is invertible precisely when $\det(A) \neq 0$, and in that case $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

- ∘ Example: For $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ with $\det(A) = 5$, $A^{-1} = \begin{bmatrix} 1/5 & -2/5 \\ 1/5 & 3/5 \end{bmatrix}$ with $\det(A^{-1}) = 1/5$.

6. The determinant of the transpose matrix is the same as the original determinant: $\det(A^T) = \det(A)$.
   - ∘ Example: $\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} = 5$ and $\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 5$ also.
7. If $A = \begin{bmatrix} | & & | & | & | & & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{i-1} & \mathbf{a}_i & \mathbf{a}_{i+1} & \cdots & \mathbf{a}_n \\ | & & | & | & | & & | \end{bmatrix}$, $B = \begin{bmatrix} | & & | & | & | & & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{i-1} & \mathbf{b}_i & \mathbf{a}_{i+1} & \cdots & \mathbf{a}_n \\ | & & | & | & | & & | \end{bmatrix}$, and $C = \begin{bmatrix} | & & | & | & | & & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{i-1} & \mathbf{c}_i & \mathbf{a}_{i+1} & \cdots & \mathbf{a}_n \\ | & & | & | & | & & | \end{bmatrix}$, where the $\mathbf{a}_j$ are column vectors and $\mathbf{a}_i = \mathbf{b}_i + \mathbf{c}_i$, then $\det(A) = \det(B) + \det(C)$. The same "linearity" property also holds for rows.
   - ∘ Example: If $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, where we decompose the second column as $\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$: then $B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. We compute $\det(A) = 1$, $\det(B) = 2$, $\det(C) = -1$, and indeed $\det(A) = \det(B) + \det(C)$.

- The most efficient way to evaluate a general determinant is via row-reduction and the definition:
  - ∘ Example: By row-reducing, $\begin{vmatrix} 1 & 2 & -1 & 3 \\ 3 & 7 & 0 & 4 \\ -2 & 1 & 1 & 2 \\ -1 & 3 & 16 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -7 \\ 0 & 5 & -1 & 8 \\ 0 & 5 & 15 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -7 \\ 0 & 0 & -16 & 43 \\ 0 & 0 & 0 & 36 \end{vmatrix} = \boxed{-576}$.

0.5 Polynomials

- Let F be a field.
- Definition: A polynomial in x with coefficients from F is an expression of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where n is a nonnegative integer and each of the coefficients a_i is in F.
  - ∘ The set of all polynomials in the variable x with coefficients from F is denoted $F[x]$.
  - ∘ Two polynomials are equal if and only if their coefficients are all equal.
- Definition: If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, and $a_n \neq 0$, we say that p is a polynomial of degree n. (The degree of the zero polynomial, whose coefficients are all zero, is defined to be $-\infty$.)
  - ∘ Example: The polynomial $p(x) = 3x^{11} - 5x^4$ has degree 11, while the polynomial $q(x) = 6$ has degree 0.
  - ∘ Observe that $\deg(pq) = \deg(p)\deg(q)$ and that $\deg(p+q) \leq \max(\deg(p), \deg(q))$.
- Polynomials can be added, subtracted, and multiplied in the usual way (using the distributive law to expand out a product): for example, $(x^2+1) + (x^3+4) = x^3 + x^2 + 5$, while $(x^2+1)(x^3+4) = x^5 + x^3 + x^2 + 4$. Division is marginally more delicate.
- Definition: We say that $a(x)$ divides $b(x)$ if there exists a polynomial $q(x)$ with $b(x) = a(x) \cdot q(x)$.
  - ∘ Example: The polynomial $x - 1$ divides $x^2 - 1$, since $x^2 - 1 = (x-1)(x+1)$.
- In a similar way to the procedure with integers, there is a "long division algorithm" for polynomials:
- Theorem (Division Algorithm): If $b(x)$ is a polynomial and $a(x)$ is a polynomial of degree $m \geq 0$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$b(x) = q(x)a(x) + r(x)$$

and where the degree of $r(x)$ is less than m.

- The idea of the algorithm is simply to perform polynomial "long division": starting with $b(x)$, subtract an appropriate multiple of $a(x)$ to remove this largest-degree term of $b(x)$, and repeat this procedure until a polynomial with degree lower than $a(x)$ is obtained.
- We will omit further details of the proof, and content ourselves instead with giving an example.

- Example: Apply the division algorithm to $a(x) = x^2 + 3x$ and $b(x) = x^3 - 5$.
  - For $a(x) = x^2 + 3x$ and $b(x) = x^3 - 5$, subtracting $xa(x)$ will remove the leading x^3 term from $b(x)$: $b(x) - x \cdot a(x) = (x^3 - 5) - x(x^2 + 3x) = -3x^2 - 5$.
  - Then adding $3a(x)$ will remove the leading $-3x^2$ from the remaining portion: $b(x) - xa(x) + 3a(x) = (-3x^2 - 5) + 3(x^2 + 3x) = -5 + 9x$.
  - The resulting polynomial now has degree less than the degree of $a(x)$, so we are done. Rearranging, we see that $b(x) = (x-3)a(x)+(-5+9x)$, so the quotient is $q(x) = x-3$ and the remainder is $r(x) = -5+9x$.

- By invoking the division algorithm with a linear polynomial, we obtain a useful property of factorization:

- Theorem (Remainder Theorem): For any a in F, the remainder obtained by dividing $p(x)$ by $x - a$ is the value $p(a)$. In particular, $x - a$ divides $p(x)$ if and only if $p(a) = 0$.
  - Proof: Since the degree of $x-a$ is 1, applying the division algorithm to $p(x)$ and $x-a$ yields a remainder $r(x)$ of degree less than 1: in other words, a constant polynomial.
  - Thus, we obtain $p(x) = (x - a)q(x) + r$. Now setting $x = a$ yields $p(a) = (a - a)q(a) + r = r$, so the remainder is equal to $p(a)$.
  - The other statement follows immediately: $x - a$ divides $p(x)$ if and only if the remainder $p(a)$ obtained by dividing $p(x)$ by $p(a)$ is equal to zero.

- Definition: If $x - r$ divides $p(x)$, we say that r is a root of the polynomial.

- When working with polynomials over the real or complex numbers, we often factor the polynomial to make it easier to study.
  - For example, over the real numbers we can write $x^4 - 1 = (x - 1)(x + 1)(x^2 + 1)$.
  - Over the complex numbers this polynomial factors further: $x^4 - 1 = (x - 1)(x + 1)(x - i)(x + i)$.

- Theorem (Fundamental Theorem of Algebra): Over the complex numbers, any polynomial can be completely factored into a product of linear terms. In other words, for any $p(x) = a_n x^n + \cdots + a_0$ with $a_n \neq 0$, there exist complex numbers $r_1, \ldots, r_n$ for which $p(x) = a_n(x - r_1)(x - r_2) \cdots (x - r_n)$.
  - Although the Fundamental Theorem of Algebra guarantees the existence of this factorization, it is often difficult to compute the roots $r_1, \ldots, r_n$.
  - If the degree of the polynomial is equal to 2, 3, or 4, there exist algebraic formulas involving radicals for the roots, but for general polynomials of larger degree, it is a theorem of Abel that no such formulas exist. (Of course, any particular polynomial of large degree might still have nice roots.)
  - For example, by completing the square, we can obtain the famous quadratic formula: the polynomial $az^2+bz+c = 0$ has the two solutions $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ over the complex numbers. When $b^2-4ac > 0$ the polynomial has two unequal real roots, when $b^2 - 4ac = 0$ the polynomial has a double real root, and when $b^2 - 4ac < 0$ the polynomial has two complex-conjugate roots.
  - However, since there are no such formulas for polynomials of large degree, unless a nice factorization happens to exist, one must use a numerical approximation procedure such as Newton's method to compute the roots.

- When factoring polynomials by hand, the following result is often useful:

- Theorem (Rational Root Test): If $p(x) = a_n x^n + \cdots + a_0$ is a polynomial with integer coefficients, and $\dfrac{a}{b}$ is a rational root (in lowest terms), then b divides a_n and a divides a_0. In particular, if the leading coefficient a_n is equal to 1, then the only possible rational roots are integers dividing the constant term a_0.

- This result follows by expanding out $p(a/b)$, clearing denominators, rearranging, and factoring appropriately: one eventually sees that b divides $a_n a^n$ and that a divides $a_0 b^n$, and since a/b is in lowest terms, b must divide a_n and a must divide a_0.
- This test cuts down on the amount of trial and error necessary for finding rational roots of polynomials, since we can write down all the possible rational roots. (Of course, a generic polynomial will not have a rational root, but our examples will usually be set up to have nice factorizations.)

- Example: Factor $p(x) = x^3 - x^2 + 4x - 4$ over the complex numbers.
  - By the rational root test, if the polynomial has a rational root then it must be an integer dividing -4: that is, one of ± 1, ± 2, ± 4. Testing the possibilities reveals that $x = 1$ is a root, and then we get the factorization $p(x) = (x-1)(x^2+4)$.
  - The roots of the quadratic (by the quadratic formula) are $x = \pm 2i$, so the full factorization is $p(x) = \boxed{(x-1)(x+2i)(x-2i)}$.

0.6 Induction

- Mathematical induction is a useful tool for proving certain kinds of results, typically, results that hold "for all positive integers".
- The principle of mathematical induction is as follows: suppose we have a sequence of statements $P(1)$, $P(2)$, $P(3)$, and so forth. If $P(1)$ is true, and $P(n)$ implies $P(n+1)$ for every $n \geq 1$, then $P(k)$ is true for every positive integer k.
  - A useful analogy for understanding the inductive principle is of climbing a ladder: if we can get on the first rung of the ladder, and we can always climb from one rung to the next, then we can eventually climb to any rung of the ladder (no matter how high).
  - We often refer to the step of showing that $P(1)$ is true as the base case, and the step of showing that $P(n)$ implies $P(n+1)$ for every $n \geq 1$ as the inductive step.

- For example, suppose we wish to show that $1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}n(n+1)$ for every positive integer n.
  - Some quick numerical experimentation will suggest that this formula is correct, but is not likely to suggest a proof.

- To prove that $1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}n(n+1)$ for every positive integer n, we can use the principle of mathematical induction.
  - If we take $P(n)$ to be the statement "$1+2+3+4+\cdots+n = \frac{1}{2}n(n+1)$", then by the inductive principle, all we need to do is show that $P(1)$ is true and that $P(n)$ implies $P(n+1)$ for each $n \geq 1$.
  - The statement $P(1)$ simply reads $1 = \frac{1}{2} \cdot 1 \cdot 2$, which is clearly true.
  - The statement $P(n)$ says that $1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}n(n+1)$, while the statement $P(n+1)$ says that $1 + 2 + 3 + 4 + \cdots + n + (n+1) = \frac{1}{2}(n+1)(n+2)$.
  - To prove that $P(n)$ implies $P(n+1)$, we need to start from the statement $1+2+3+4+\cdots+n = \frac{1}{2}n(n+1)$ and use it (somehow) to show that $1 + 2 + 3 + 4 + \cdots + n + (n+1) = \frac{1}{2}(n+1)(n+2)$.
  - We can do this as follows: observe that

$$\begin{aligned} 1 + 2 + 3 + 4 + \cdots + n + (n+1) &= [1 + 2 + 3 + 4 + \cdots + n] + (n+1) \\ &= \frac{1}{2}n(n+1) + (n+1) = \frac{1}{2}(n^2 + 3n + 2) = \frac{1}{2}(n+1)(n+2). \end{aligned}$$

where we applied the "inductive hypothesis" piece of information that $1+2+3+4+\cdots+n = \frac{1}{2}n(n+1)$ to go from the first line to the second, and then simply did algebra to rearrange the result into the desired expression.

- Since we have proven the two required pieces, namely that $P(1)$ is true and that $P(n)$ implies $P(n+1)$, by the principle of mathematical induction, $P(k)$ is true for every $k \geq 1$.

- Induction arguments are useful because they can convert difficult direct proofs into (often) comparatively routine exercises.
  - The base case is usually an easy example where the result is obvious or almost obvious, while the inductive step gives a clear hypothesis to start with and an equally clear goal to reach. Generally, most of the work in the proof goes into the proof of the inductive step.

- Here are a few more examples of proofs by induction, written in a more typical style.

- Example: Prove that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for every positive integer n.
  - We prove this result by induction on n.
  - For the base case $n = 1$, we must show that $1 = 1$ which is clearly true.
  - For the inductive step, we are given that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ and must show that $1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = (n+1)^2$.
  - By the inductive hypothesis, we can write

    $$1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = [1 + 3 + 5 + \cdots + (2n-1)] + (2n+1)n^2 + 2n + 1 = (n+1)^2$$

    and therefore we see $1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = (n+1)^2$, as required.
  - By induction, $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for every positive integer n.

- Example: Prove that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ for every positive integer n.
  - We prove this result by induction on n.
  - For the base case $n = 1$, we must show that $\frac{1}{1\cdot 2} = \frac{1}{2}$ which is clearly true.
  - For the inductive step, we are given that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ and must show that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} + \frac{1}{(n+1)\cdot(n+2)} = \frac{n+1}{n+2}$.
  - By the inductive hypothesis, we can write

    $$\begin{aligned} \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} + \frac{1}{(n+1)\cdot(n+2)} &= \left[\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)}\right] + \frac{1}{(n+1)\cdot(n+2)} \\ &= \frac{n}{n+1} + \frac{1}{(n+1)\cdot(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2} \end{aligned}$$

    and therefore we see $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} + \frac{1}{(n+1)\cdot(n+2)} = \frac{n+1}{n+2}$, as required.
  - By induction, $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ for every positive integer n.

- There are various modifications to this "basic" form of induction. The procedure for any induction problem is essentially the same, however: we establish a base case, and prove an inductive step.
  - We often want to start at a different base case than $n = 1$: frequently, we instead start at $n = 0$ or $n = 2$.
  - As long as we establish the appropriate base case and inductive step, the inductive principle still works.

- If, for example, our base case is $n = 2$, then we would prove $P(2)$ is true and that $P(n)$ implies $P(n+1)$, with the conclusion being that $P(k)$ is true for all integers $k \geq 2$.

- Example: Show that $2^n > n^2$ for all integers $n \geq 5$.
  - We prove this result by induction on n.
  - For the base case $n = 5$, we must show that $2^5 > 5^2$, or $32 > 25$, which is clearly true.
  - For the inductive step, we are given that $2^n > n^2$ and $n \geq 5$, and must show that $2^{n+1} > (n+1)^2$.
  - By the inductive hypothesis, we can write $2^{n+1} = 2 \cdot 2^n > 2n^2$.
  - Furthermore, since $n \geq 5$, $2n^2 = n^2 + n^2 \geq n^2 + 5n \geq n^2 + 2n + 1 = (n+1)^2$.
  - Putting the inequalities together, we see that $2^{n+1} > 2n^2 \geq (n+1)^2$, so $2^{n+1} > (n+1)^2$ as required.
  - Therefore, by induction, $2^n > n^2$ for all integers $n \geq 5$.

- Another flavor of induction is called "complete induction" or "strong induction": rather than assuming the immediately previous case, we assume *all* of the previous cases: the inductive step is now that $P(1)$, $P(2)$, ..., $P(n)$ collectively imply $P(n+1)$.
  - It may seem like we are assuming extra information, but in fact strong induction and (regular) induction are logically equivalent.

- Example: Prove that every positive integer $n \geq 2$ can be factored into a product of prime numbers, where we consider a product involving only one term to be a product. (Recall that an integer greater than 1 is prime when its only positive divisors are 1 and itself.)
  - We prove this result by strong induction on n.
  - For the base case, we take $n = 2$, which is already prime.
  - For the inductive step, suppose $n \geq 3$ and that all integers less than n can be factored into a product of prime numbers.
  - If n is prime, then we are already done. Otherwise, we may factor $n = ab$ for some $1 < a, b < n$.
  - Then a and b are both between 1 and n, and so by the strong induction hypothesis, both a and b can be written as a product of prime numbers. Hence, so can n: simply multiply the two products.
  - In all cases, n can be factored into a product of prime numbers, which establishes the inductive step.
  - Since both the base case and inductive step hold, by strong induction, every positive integer $n \geq 2$ can be factored into a product of prime numbers.

- In some situations, it may be necessary to have multiple base cases depending on the structure of the argument:

- Example: Let $a_0 = 2$, $a_1 = 5$, and, for $n \geq 2$, let $a_n = 5a_{n-1} - 6a_{n-2}$. Prove that $a_n = 2^n + 3^n$ for all $n \geq 0$.
  - We prove this result by strong induction on n.
  - For $n = 0$ and $n = 1$, the result is obvious, since $a_0 = 2 = 2^0 + 3^0$ and $a_1 = 5 = 2^1 + 3^1$.
  - Now suppose $n \geq 2$. By the strong induction hypothesis and the fact that $n \geq 2$, we have $a_{n-1} = 2^{n-1} + 3^{n-1}$ and $a_{n-2} = 2^{n-2} + 3^{n-2}$, and we want to show that $a_n = 2^n + 3^n$.
  - By the recursion and the induction hypotheses,

$$\begin{aligned} a_n &= 5a_{n-1} - 6a_{n-2} \\ &= 5(2^{n-1} + 3^{n-1}) - 6(2^{n-2} + 3^{n-2}) \\ &= 4 \cdot 2^{n-2} + 9 \cdot 3^{n-2} = 2^n + 3^n \end{aligned}$$

  and therefore $a_n = 2^n + 3^n$ as claimed. By (strong) induction, we conclude that $a_n = 2^n + 3^n$ for all integers $n \geq 0$.

Well, you're at the end of my handout. Hope it was helpful.