Dynamics, Chaos, and Fractals (part 1): Introduction to Dynamics (by Evan Dummit, 2015, v. 1.07)

Contents

1 Introduction to Dynamics

In this chapter, our goal is to provide an introduction to (discrete) dynamical systems on the real line. We begin by introducing some examples of more general dynamical systems, and then restrict our attention to discrete dynamical systems on $\mathbb{R}$, which arise by repeatedly iterating a function f defined on a subset of the line. We study the structure of the orbit of a point under a function, and how to classify the behavior of fixed points and cycles in terms of whether they attract or repel nearby points as we iteratively apply f. We will then discuss Newton's method, a procedure often familiar from calculus that provides a way to compute zeroes of differentiable functions numerically, both because it is a computational aid and because it provides another source of interesting dynamical systems.

1.1 Examples of Dynamical Systems

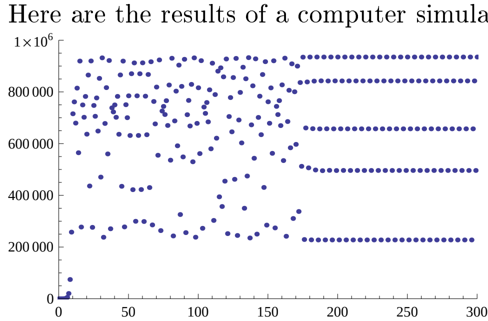
- Vaguely speaking, a dynamical system consists of two things: a space, and an operator that acts on that space.
- In a discrete dynamical system, the operator can be thought of as a transformation of the space. The goal is to describe the behavior of points in the space as we iterate the transformation repeatedly.
  - Example: The space is $\mathbb{R}$ and the transformation is the function $f(x) = \cos(x)$. What happens as we apply f repeatedly to a particular value of x (e.g., $x = 2\pi$)?
  - Example: The space is $\mathbb{R}$ and the transformation is the function $f(x) = x^2 - 1$. What happens as we apply f repeatedly?
  - Example: The space is $\mathbb{C}$ and the transformation is the function $f(z) = \dfrac{z+i}{z-i}$. What happens as we apply f repeatedly?
  - Example: The space is $\mathbb{R}^3$ and the transformation is the map $T(x, y, z) = (x + 2y, \sin(z), \cos(x))$. What happens as we apply T repeatedly?

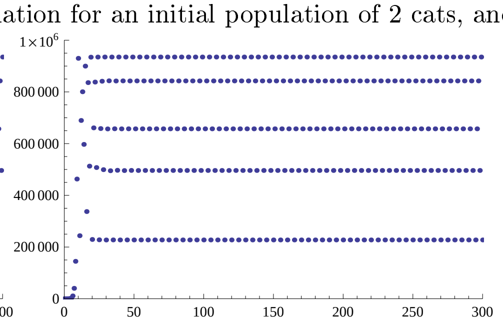
- In a continuous dynamical system, the operator can be thought of as a set of rules that tell the system how to change (or "evolve") over time.
  - Any differential equation, such as $f''(t) = e^{2t} f(t) + f'(t)$, gives a dynamical system (in the sense above) – the goal is to solve the equation, or at least describe its solutions.
  - Another example: consider a solar system containing 2 planets with some given initial positions and velocities. As time goes forward, what kind of motion will the planets have? (This is called the 2-body problem, and it was essentially solved by Newton.)
  - More generally, consider a solar system with n planets and given initial information. What kind of motion will the planets have? (This is called the n-body problem, and, perhaps surprisingly, its solutions can behave very unpredictably, unlike the solutions to the 2-body problem.)
  - Earth's climate is another example of a dynamical system: it is well known that predicting the weather into the future with high accuracy is very computationally difficult.
  - Similar computational difficulties often arise in many other common models of physical phenomena. Since any physical phenomenon that evolves over time gives rise to a dynamical system, it is worthwhile to study abstract properties of dynamical systems.

- We will see later that even simple-seeming dynamical systems can exhibit extremely complicated and unpredictable "chaotic" behavior. But we will give a taste of what is to come by analyzing the dynamics of a few simple population models:

- <u>Example</u> (population model 1): A population of cats lives on an extremely large desert island with plentiful food. When the population is small, the cats can essentially breed with no restrictions. If the population is currently P, then the population one year later will be $4P$ (with one pair of cats producing eight offspring per year on average).
  - In this case, the population at arbitrary number of years later can be found by iterating the function $f(P) = 4P$: $f(f(P)) = 16P$, $f(f(f(P))) = 64P$, and so forth.
  - Thus, if the population at year 0 is 2 cats, then after n years, there will be $4^n \cdot 2$ cats: i.e., we observe exponential population growth.
  - If we change the starting population or the growth parameter slightly, the system will behave very predictably over time: we will always get exponential growth (at least, assuming the growth constant actually makes the population increase).

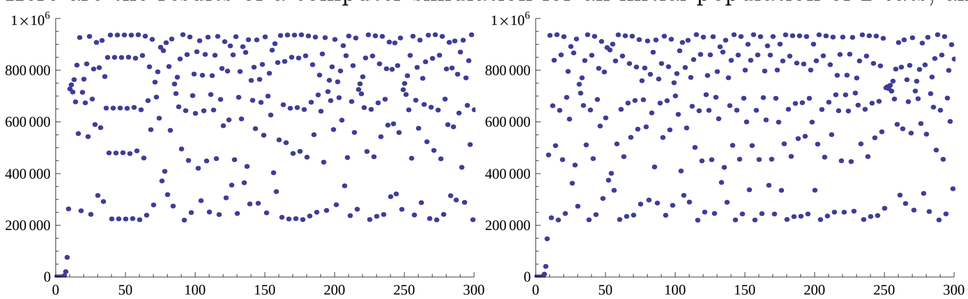
- <u>Example</u> (population model 2): On another much smaller desert island there is also a population of cats. Since this island is smaller, when the population grows sufficiently large the cats will begin to compete for resources and breed more slowly (or even decrease in population, if there are more cats than the island can sustain). After careful study, it is determined that if the population is currently P, then the population one year later will be $3.74P\left(1 - \frac{P}{1\,000\,000}\right)$.
  - In this case, the population at arbitrary number of years later can be found by iterating the function $f(P) = 3.74P\left(1 - \frac{P}{1\,000\,000}\right)$.
  - Here are the results of a computer simulation for an initial population of 2 cats, and of 4 cats:





- After a few generations of nearly-exponential growth, the population for a starting population of 2 cats seems to bounce around randomly for about 170 generations, but then settles into an extremely stable pattern that oscillates between five population values which are (in order of how they appear from year to year) equal to 227476, 657233, 842539, 496175, and 934945.
- For 4 cats, the behavior is very similar (and the cycling population values are the same), but the pattern stabilizes much more quickly, after about 30 generations.

- Example (population model 3): On a very slightly different desert island to the one considered above, the cats breed a very tiny bit more rapidly: if the current population is P, then the next year's population is $3.75P\left(1 - \frac{P}{1\,000\,000}\right)$.
  - Here are the results of a computer simulation for an initial population of 2 cats, and of 4 cats:



  - Unlike the previous example, the populations both appear to behave in a much less predictable manner. There are some runs where the populations almost cycle between a small number of values, but they are not stable and degenerate into seemingly random behavior quite rapidly.
  - Later, we will study the dynamics of these types of maps, and give some explanations for the radically different behaviors of these two seemingly similar models.

1.2 Orbits, Fixed Points, and Cycles of Periodic Points

- Our primary aim is to study the following question: given a function f defined on a subset of the real line $\mathbb{R}$, and a point x_0, describe the behavior of the sequence x_0, $f(x_0)$, $f(f(x_0))$, $f(f(f(x_0)))$, ...

- Notation: Since we will frequently speak of the iterate of a function, we define $f^n(x)$ to be the result of iterating f a total of n times on x. Thus, $f^1(x) = f(x)$, $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$, and in general, $f^n(x) = f(f^{n-1}(x))$ for any $n \geq 2$. We also adopt the convention that $f^0(x) = x$, the result of applying f "zero times".
  - This conflicts with the convention, often used elsewhere, that the expression $\sin^2(x)$ is to be interpreted as $[\sin(x)]^2$. In these notes we will therefore avoid the iterated function notation with explicitly-written trigonometric functions, and write explicitly that a function is being squared (when such a thing occurs).
  - Do not confuse the iterated function notation with the notation for higher-order derivatives: $f^3(x)$ means the triple iterate $f(f(f(x)))$, while $f^{(3)}(x)$ means the third derivative $f'''(x)$.

1.2.1 Definitions and Examples

- Definition: The orbit of x_0 under f is the sequence x_0, x_1, x_2, x_3, ... where $x_n = f^n(x_0)$. The value x_0 is called the seed or initial point of the orbit.

- Example: Describe the orbits of $x_0 = 2$, 0, 1, and $\frac{1}{2}$ under the function $f(x) = x^2$.
  - For $x_0 = 2$, the orbit is 2, 4, 16, 256, 65536, and so forth. This orbit clearly grows to ∞.
  - For $x_0 = 0$, the orbit is 0, 0, 0, 0, 0, 0, and so forth. This orbit remains fixed at 0.
  - For $x_0 = 1$, the orbit is 1, 1, 1, 1, 1, 1, and so forth. This orbit remains fixed at 1.

- For $x_0 = \frac{1}{2}$, the orbit is $\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \frac{1}{65536}$, and so forth. This orbit approaches the limiting value 0.

- Example: Describe the orbits of $x_0 = 2$, 0, 1, and $\frac{1}{2}$ under the function $f(x) = x^2 - 1$.
  - For $x_0 = 2$, we get the orbit 2, 3, 8, 63, 3968, and so forth. This orbit clearly grows to ∞.
  - For $x_0 = 0$, we get the orbit 0, -1, 0, -1, 0, -1, and so forth. The values will clearly continue cycling between 0 and -1 as we continue applying f.
  - For $x_0 = 1$, we get the orbit 1, 0, -1, 0, -1, 0, -1, and so forth. These values likewise will continue cycling between 0 and -1.
  - For $x_0 = \frac{1}{2}$, we get the values (to four decimal places) 0.5, -0.75, -0.4375, -0.8086, -0.3462, -0.8802, -0.2253, -0.9492, -0.0990, -0.0195, -0.9996, -0.0008. As we continue applying f, the values are clearly approaching -1 and 0.

- The above examples demonstrate a number of typical orbit behaviors, which we now define:

- Definition: A fixed point of a function $f(x)$ is a point x_0 such that $f(x_0) = x_0$.
  - Since finding fixed points is equivalent to solving the equation $f(x) = x$, we can qualitatively search for a function's fixed points by drawing the graphs of $y = f(x)$ and $y = x$ and looking for intersection points.
  - Example: The function $f(x) = x^2$ has two fixed points, namely $x = 0$ and $x = 1$, since the solutions to $x^2 = x$ are $x = 0$ and $x = 1$.
  - Example: The function $f(x) = x + 1$ has no fixed points, because there are no values of x satisfying $x + 1 = x$.
  - Example: The function $f(x) = x\cos(\pi x)$ has infinitely many fixed points, namely $x = 2k$ for any integer k: solving $x\cos(\pi x) = x$ produces $x = 0$ or $\cos(\pi x) = 1$, and the solutions to the latter are $\pi x = 2\pi k$ for an integer k.

- Definition: A value x_0 is called a periodic point for f, and its orbit is called a periodic orbit (or an n-cycle), if there is some value of n such that $f^n(x) = x$. Any such value of n is called a period of x_0, and the smallest (positive) value of n is called the minimal period (or exact period).
  - A periodic orbit of length n will repeat every n steps: it is x_0, $f(x_0)$, $f^2(x_0)$, ... , $f^{n-1}(x_0)$, x_0, $f(x_0)$, $f^2(x_0)$, ...
  - Notice by definition that if x_0 is periodic with period n, then so is $f^k(x_0)$ for any k (since their orbits will all cycle through the same n values). Also by definition, x_0 is a periodic point of period n precisely when x_0 is a fixed point of f^n, since both statements say that $f^n(x_0) = x_0$.
  - Example: Any fixed point of a function is a periodic point of exact period 1.
  - Example: The point $x_0 = -1$ is a periodic point of period 2 for the function $f(x) = x^2 - 1$, since $f(-1) = 0$ and $f^2(-1) = -1$. Likewise, 0 is also a periodic point of period 2 for $f(x)$.
  - Example: The point $x_0 = 1$ is a periodic point of period 3 for the function $f(x) = 1 - \frac{1}{2}x - \frac{3}{2}x^2$, since $f(1) = -1$, $f^2(1) = 0$, and $f^3(1) = 1$.
  - Example: The point $x_0 = 1$ is a periodic point of period 4 for the function $f(x) = \sqrt{2} - \frac{1}{x}$, since $f(1) = \sqrt{2} - 1$, $f^2(1) = -1$, $f^3(1) = \sqrt{2} + 1$, and $f^4(1) = 1$.
  - Note: Some authors use the term "prime period" for the minimal period. This is somewhat misleading, because the length of the minimal period need not be a prime number, as the previous example shows.

- We record here a pair of basic facts about periodic points:

- Proposition: If x_0 is a periodic point with minimal period n, then $f^m(x_0) = f^{m+n}(x_0)$ for any m, and $f^k(x_0) = x_0$ holds if and only if n divides k.

- Proof: For the first statement, simply apply f^m to both sides of the statement $f^n(x_0) = x_0$. For the forward direction of the second statement, setting $m = dn$ for $d = 1, 2, 3$, yields $x_0 = f^n(x_0) = f^{2n}(x_0) = f^{3n}(x_0) = \cdots$, so $f^k(x_0) = x_0$ if k is a multiple of n.
- For the reverse direction, suppose that n is the minimal period of x_0 and that $f^k(x_0) = x_0$ but n does not divide k, so that $k = qn + r$ for some integer q and some integer r with $0 < r < n$. By the definition of the period, we have $f^r(x_0) = f^{n+r}(x_0) = f^{2n+r}(x_0) = \cdots = f^{qn+r}(x_0) = f^k(x_0) = x_0$, but this is a contradiction because then r is a period for x_0 that is smaller than n.

- Finding all the periodic points of order n for f requires solving $f^n(x) = x$. For most functions this is computationally quite difficult: if f is a polynomial of degree d, then $f^n(x) - x$ is a polynomial of degree d^n.
  - For polynomials, we only have a hope of doing this (even with a computer) if f is a polynomial of small degree and the order is small.
  - However, if $m|n$, then every point of period m will satisfy $f^m(x) = x$ and hence also $f^n(x) = x$. Thus, we can save a small amount of effort by removing the factors of $p^n(x) - x$ that come from terms $p^m(x) - x$ where $m|n$.
  - This trick is especially helpful if f is a quadratic polynomial and $n = 2$: then $f^2(x) - x$ has degree 4, but it is divisible by the quadratic $f(x) - x$, so we can take the quotient and obtain a quadratic, which is then easy to solve.

- Example: Determine the values of λ, with $0 < \lambda \leq 4$, for which the logistic map $p_\lambda(x) = \lambda x(1 - x)$ has a real-valued 2-cycle.
  - By the remarks above, $p_\lambda^2(x) - x$, whose zeroes are the points of period 1 or 2 for p_λ, is necessarily divisible by $p_\lambda(x) - x$, whose zeroes are the points of period 1 (by properties of polynomials).
  - Some algebra shows that $p_\lambda^2(x) - x = -\lambda^3 x^4 + 2\lambda^3 x^3 - (\lambda^2 + \lambda^3)x^2 + (-1 + \lambda^2)x$, so dividing it by $p_\lambda(x) - x = -\lambda x^2 + (-1 + \lambda)x$ yields the quotient $q(x) = \lambda^2 x^2 - (\lambda + \lambda^2)x + (1 + \lambda)$.
  - We can straightforwardly compute that the roots of q are $r_1, r_2 = \dfrac{1 + \lambda \pm \lambda\sqrt{\lambda^2 - 2\lambda - 3}}{2\lambda}$.
  - If $\lambda^2 - 2\lambda - 3$ is negative (which on the given range occurs whenever $\lambda < 3$), there are no real-valued solutions and hence no real-valued 2-cycle.
  - If $\lambda = 3$, then the fixed points are $x = 0$ and $x = \dfrac{2}{3}$, while the double root of the quadratic q is $r = \dfrac{2}{3}$. So in this case, we do not get a 2-cycle (instead, one of the fixed points shows up repeatedly).
  - If $3 < \lambda \leq 4$, then we get a 2-cycle: the polynomial p_λ interchanges the two real roots r_1 and r_2 given above. So, we get a 3-cycle precisely when $\boxed{3 < \lambda \leq 4}$.

- The last kind of behavior is where an orbit is not periodic, but eventually falls into a repeating cycle.

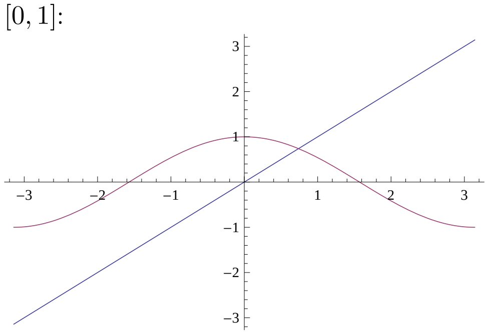
- Definition: A value x_0 is called a preperiodic point for f (or eventually periodic) if there exist positive integers m and n such that $f^m(x_0) = f^{m+n}(x_0)$. Equivalently, x_0 is preperiodic if there exists some m so that $f^m(x_0)$ is periodic. In the event that $n = 1$, we say x_0 is an eventually fixed point.
  - Example: The point $x_0 = 1$ is a preperiodic point for the function $f(x) = x^2 - 1$, since the orbit of 1 is 1, 0, -1, 0, -1, 0, -1, ...
  - Example: The point $x_0 = -1$ is an eventually fixed point for the function $f(x) = x^2$, since the orbit of -1 is -1, 1, 1, 1, 1, 1, ...
  - Example: The point $x_0 = \dfrac{1}{3}$ is a preperiodic point for the function $f(x) = 1 - \dfrac{1}{2}x - \dfrac{3}{2}x^2$, since the orbit of $\dfrac{1}{3}$ is $\dfrac{1}{3}$, $\dfrac{2}{3}$, 0, 1, -1, 0, ...

1.2.2 General Existence of Fixed and Periodic Points

- For complicated functions, it is not possible to solve for fixed points or periodic points exactly.
  - For example, a graph will indicate that $f(x) = \cos(x)$ has a fixed point, but it is not possible to solve the equation $x = \cos(x)$ algebraically.
  - Solving for periodic points causes similar problems, even with polynomials of small degree. For example, if $f(x) = x^2 - 1$, then looking for periodic points of period 3 requires solving the degree-8 equation $f(f(f(x))) = x$. It turns out that there are no real periodic points of period exactly 3 for this function, but this is not at all easy to see!

- One way that we can show the existence of a fixed point (or a periodic point, since that is the same as a fixed point of f^n) is using the Intermediate Value Theorem.
  - Recall that the Intermediate Value Theorem says that if $f(x)$ is a continuous function on the interval $[a, b]$, then for any value y between $f(a)$ and $f(b)$, there is some value of c in (a, b) such that $f(c) = y$.
  - To show the existence of a fixed point of a continuous function f, we want to invoke the Intermediate Value Theorem to the function $g(x) = f(x) - x$ to show that g takes the value zero. For this, it is enough to find one value where g is negative and another where g is positive: then g must be zero somewhere in between, and this location is a fixed point of f.

- Example: Show that $f(x) = \cos(x)$ has a fixed point.
  - By looking at a graph, we can see that $y = \cos(x)$ and $y = x$ intersect once, somewhere in the interval $[0, 1]$:



  - If $g(x) = f(x) - x$, then $g(0) = 1$ while $g(\pi/2) = -\pi/2$, so since cosine is continuous, the function $\cos(x)$ has a fixed point in the interval $(0, \pi/2)$.
  - By using the graph to get a better guess for the interval where the fixed point lies, or using more intelligent root-finding algorithms such as Newton's method, we can rapidly approximate the value of the fixed point. In this case, to six decimal places, the value of the fixed point is 0.739085.

- Example: Show that $f(x) = x^3 - 3x$ has a periodic point of order 2 lying in the interval $(1, 1.5)$ and a periodic point of order 3 lying in the interval $(0.4, 0.5)$.
  - The idea for the point of order 2 is to show that $g(x) = f^2(x) - x$ has a root in this interval but that $f(x) - x$ does not have a root in this interval: the first statement will imply the existence of a periodic point of order dividing 2, and the second will imply it cannot have order 1.
  - Similarly, for the point of order 3, we want to show that $h(x) = f^3(x) - x$ has a root in the interval $(0.4, 0.5)$ but that $f(x) - x$ does not.
  - Since $f(x) - x = x^3 - 4x$ has roots $x = 0, \pm 2$, it does not have roots in either interval.
  - Now we compute $g(1) = f^2(1) - 1 = -3$ and $g(1.5) = 0.451$, so we conclude that g must be zero in this interval, and thus that f has a periodic point of order 2. (In fact, one can show that this periodic point is $x_0 = \sqrt{2}$.)
  - Similarly, $h(0.4) = 1.098$ but $h(0.5) = -1.527$, so h has a zero in this interval and f has a periodic point of order 3. (Solving for its exact value would require factoring the polynomial $h(x)$, which has degree 27. Not an easy task!)

- An application of the Intermediate Value Theorem is to prove that any continuous function that maps some interval into itself must have a fixed point:

- Proposition: If $f : [a, b] \to [a, b]$ is a continuous function, then it has a fixed point.
  - Note that the function need not be surjective (i.e., it does not need to have every point of $[a, b]$ in its image) – its image just needs to be contained in $[a, b]$.
  - Remark: This is the 1-dimensional case of a much more general theorem known as the Brouwer Fixed-Point Theorem, one version of which states that any continuous function from a closed, bounded, convex subset of $\mathbb{R}^n$ to itself must have a fixed point.
  - Proof: Let $g(x) = f(x) - x$: we have $g(a) = f(a) - a \geq 0$ since $f(a) \in [a, b]$, and we also have $g(b) = f(b) - b \leq 0$ since $f(b) \in [a, b]$. Applying the Intermediate Value Theorem to $g(x)$ on $[a, b]$ shows that g has a zero in $[a, b]$, which is the desired fixed point of f.

- A natural question at this point is: what kinds of orders of periodic points can occur for a given function? We will return to this question much later, but we will give a few examples illustrating different kinds of behaviors:
  - A function $f(x)$ need not have any fixed or periodic points at all: for example, $f(x) = x + 1$ has no fixed points nor any periodic points, since $f^n(x) = x + n$ is clearly never equal to x for any $n > 0$.
  - A function can have infinitely many fixed points: an example is $f(x) = x + \sin(x)$.
  - A periodic point can have any given order: for example, there exists a polynomial of degree n which sends 0 to 1, 1 to 2, 2 to 3, ... , $n - 1$ to n, and n to 0.
  - Every point in the domain of a function can be a periodic point: an example is $f(x) = a - x$ for any constant a.
  - It might seem as if there are no restrictions on what kinds of behaviors can occur, but this turns out not to be the case.

- Proposition: If $f(x)$ is a continuous function that has no fixed points, then f has no periodic or preperiodic points at all.
  - Proof: If $f(x) - x$ is a continuous function that is never zero, then it must either be always positive or always negative. Suppose it is always positive: then $f(x) > x$ for all x. But then $f^2(x) > f(x) > x$ for all x, and, iterating, we see that $f^3(x) > f^2(x) > x$, $f^4(x) > f^3(x) > x$, and in general, $f^n(x) > x$ for any positive n. Thus, f cannot have any periodic points, or any preperiodic points. We get a similar contradiction if $f(x) - x$ is always negative, by the same argument with all of the inequalities reversed.

1.2.3 The Doubling Function, the Logistic Maps, and Computational Difficulties

- It is tempting to believe that, although we cannot necessarily solve for fixed points and periodic points algebraically, if we simply use a computer with high enough accuracy, we will be able to study orbit behaviors with no difficulty.

- However, there are a number of simple functions that demonstrates the fallacy of this belief.

- Definition: The doubling function $D : [0, 1) \to [0, 1)$ is defined as $D(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x < 1 \end{cases}$. Equivalently, $D(x)$ is the residue of $2x$ modulo 1 (i.e., the result obtained by removing the "integer part" of $2x$).

- It is simple to analyze orbits of rational numbers under the doubling function using exact arithmetic.
  - Example: The orbit of 0 is 0, 0, 0, 0, ..., which is a fixed point. It is easy to see that 0 is the only fixed point for D.
  - Example: The orbit of $\frac{1}{3}$ is $\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}$, ..., which is a 2-cycle.
  - Example: The orbit of $\frac{3}{7}$ is $\frac{3}{7}, \frac{6}{7}, \frac{5}{7}, \frac{3}{7}, \frac{6}{7}, \frac{5}{7}$, ..., which is a 3-cycle.

- Example: The orbit of $\frac{1}{8}$ is $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}$, 0, 0, 0, ..., so $\frac{1}{8}$ is an eventually fixed point.

- Indeed, there are a number of elementary facts one can make about the orbits of the doubling function.
  - For example, every rational number $\frac{p}{q}$ is a preperiodic point for f: it is easy to see that $D\left(\frac{p}{q}\right)$ is also a rational number in $[0,1)$ with denominator q, and there are only q such rational numbers, so eventually they must start repeating.
  - More specifically, any rational number with odd denominator is actually a periodic point for f: the function D is a bijection from the set $\left\{\frac{0}{q}, \frac{1}{q}, \frac{2}{q}, \cdots, \frac{q-1}{q}\right\}$ to itself if q is odd, since $T(a) = T(b)$ can happen only if a and b are equal or differ by $\frac{1}{2}$, and $\frac{1}{2}$ cannot be written as $\frac{p}{q}$ with q odd.
  - Any rational number with an even denominator (in lowest terms), on the other hand, will be a strictly preperiodic point, since $T\left(\frac{p}{q}\right)$ will have denominator $q/2$ if q is even.
  - The converse is also true: every preperiodic point for f is a rational number. This follows first by observing (by a trivial induction) that $D^n(x) = 2^n x$ modulo 1: so if x is a preperiodic point with $D^{m+n}(x) - D^m(x) = 0$, then $2^{m+n}x - 2^m x$ is congruent to 0 modulo 1. In other words, $(2^{m+n} - 2^m)x$ is an integer, meaning that x is rational.

- However, if we try to use a decimal approximation to analyze the orbits of D, we will get very erroneous results:
  - Suppose we try to describe the orbit of $\frac{1}{3}$ by using different decimal approximations of $\frac{1}{3}$.
  - Here is a table of the orbits of two decimal approximations:

Term	0	1	2	3	4	5	6	7	8	9	10	11
$x_0 = 1/3$	1/3	2/3	1/3	2/3	1/3	2/3	1/3	2/3	1/3	2/3	1/3	2/3
$x_0 = 0.33$	0.33	0.66	0.32	0.64	0.28	0.56	0.12	0.24	0.48	0.96	0.92	0.84
$x_0 = 0.333$	0.333	0.666	0.332	0.664	0.328	0.656	0.312	0.624	0.248	0.496	0.992	0.984

  - Notice that the first few elements of each approximate orbit are fairly close to the correct values. But after 10 iterations, the orbits of 0.33 and 0.333 have wandered quite far away from the orbit of 1/3, and from each other.
  - Increasing the accuracy of the decimal approximation will not help significantly, either: if ϵ is small and x and $x + \epsilon$ are both either in $(0, 1/2)$ or $(1/2, 1)$, then it is easy to check that $D(x + \epsilon) - D(x) = 2\epsilon$. Thus, each iteration will double the error (at least, until $D^n(x + \epsilon)$ and $D^n(x)$ are sufficiently far apart).

- Of course, for rational numbers, it is easy to use exact rational arithmetic, as we did above. But what can be done to study orbits of irrational points under the map D?
  - For example, how would one compute $D^{20}(\sqrt{2} - 1)$ to three decimal places?
  - We would ultimately need to use a decimal approximation of $\sqrt{2} - 1$ at some stage, but we would need to determine the proper number of decimal places to carry out computations to, in order to ensure that we do not lose too much accuracy by iterating the map D.
  - Such calculations become increasingly computationally expensive as we travel further out in the orbit, since we will need to keep finding better decimal approximations as we continue.
  - There is something fundamental about the doubling function that resists numerical computation: it is "sensitive to initial conditions". (We will postpone further discussion of these issues until we define chaotic functions in a later chapter.)

- Another class of examples that cause computational problems are the logistic maps $p_\lambda(x) = \lambda x(1 - x)$, for a fixed parameter $0 < \lambda \le 4$. (The bound on λ is so that p_λ is a map from $[0, 1] \to [0, 1]$.)
  - Let us attempt to compute the orbit of $\frac{1}{3}$ under the map $p_4(x) = 4x(1 - x)$: the first six terms are $\frac{1}{3}$, $\frac{8}{9}, \frac{32}{81}, \frac{6272}{6561}, \frac{7250432}{43056721}$, and $\frac{1038154236987392}{1853020188851841}$.

- Clearly, using rational arithmetic is not going to be computationally efficient, because the number of digits in both the numerator and denominator will double at every stage. It is fairly easy to show that the denominator of $f^n(1/3)$ is 3^{2^n}, but there is not a nice formula for the numerators.
- Here is a table of a computation of the orbit of $\frac{1}{3}$ under this map, where each step's computation was rounded to the stated number of decimal places. (The results are stated to 4 decimal places so as not to make the table too large, but the computations retained the stated amount of data.):

Term	0	1	2	3	4	5	10	20	30	50
4 places	0.3333	0.8889	0.3950	0.9559	0.1686	0.5607	0.8669	0.5655	0.9558	0.4139
8 places	0.3333	0.8889	0.3951	0.9560	0.1684	0.5603	0.8747	0.0158	0.1025	0.3208
15 places	0.3333	0.8889	0.3951	0.9560	0.1864	0.5603	0.8747	0.0163	0.7531	0.8049

- As should be clear, the first few terms are stable with only a few digits, but the computations diverge rapidly after 30 or so iterations of f.
- We can see, then, that it is a nontrivial problem in numerical analysis to determine the needed accuracy to ensure that the orbit calculations are accurate, even for this simple quadratic polynomial.

1.3 Qualitative and Quantitative Behavior of Orbits

- We would like to describe orbits in a more precise way than "plug in some values and hope it's possible to guess what happens". There are a number of different approaches, some geometric, some algebraic.

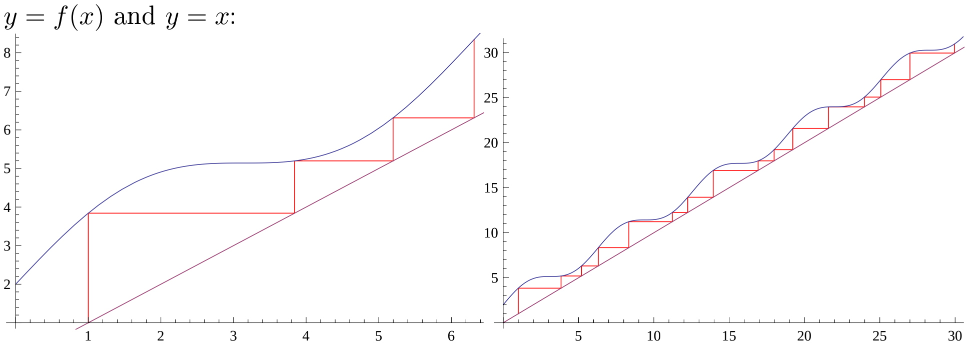
1.3.1 Orbit Analysis Using Graphs

- One way we can study the orbit of x_0 when we iterate a function f is by using the graph of the function $y = f(x)$, in the following manner: first, we plot $y = f(x)$ and $y = x$, and the initial point (x_0, x_0). Then we alternate the following two steps to create a "staircase":

  1. Drawing a vertical line from the current point to the intersection with $y = f(x)$, and
  2. Drawing a horizontal line from the current point to the intersection with $y = x$.

  This will construct the sequence of points (x_0, x_0), $(x_0, f(x_0))$, $(f(x_0), f(x_0))$, $(f(x_0), f^2(x_0))$, $(f^2(x_0), f^2(x_0))$, $(f^2(x_0), f^3(x_0))$, ..., whose coordinates describe the orbit of x_0 under f.

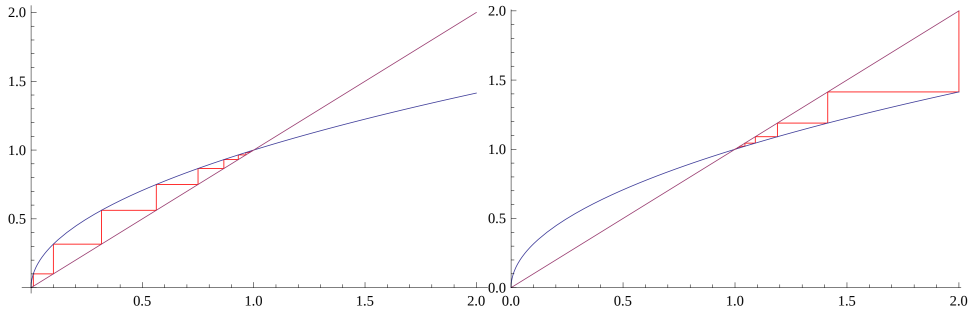
- <u>Example</u>: Plot the orbit of $x_0 = 1$ for the function $f(x) = x + \sin(x) + 2$.
  - Here are plots of the "staircase" after 4 and 16 iterations (respectively) of f, along with the graphs of $y = f(x)$ and $y = x$:



  - This function was intentionally chosen so that it would always lie above $y = x$ (in order to emphasize the "staircase" behavior).
  - We can see that the orbit of $x_0 = 1$ will blow up to ∞, since it will continue moving to the right as we continue iterating.

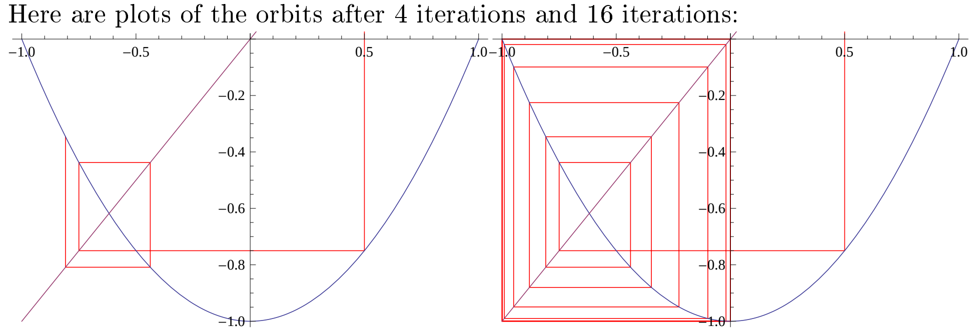
- <u>Example</u>: Plot the orbits of $x_0 = 0.01$ and $x_0 = 2$ for the function $f(x) = \sqrt{x}$.

- We obtain the following pictures for the two respective orbits:



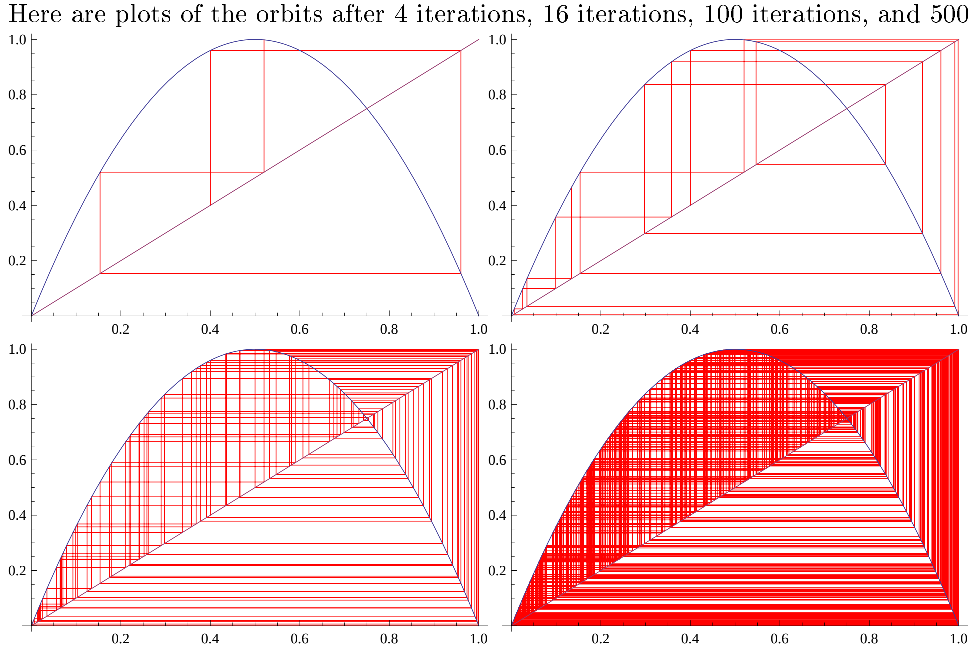
- Note that the orbit of 0.01 travels to the right, while the orbit of 2 travels to the left. We can see quite clearly that both orbits are moving toward the fixed point $x_0 = 1$.

- Example: Plot the orbit of $x_0 = \dfrac{1}{2}$ for the function $f(x) = x^2 - 1$.
  - Here are plots of the orbits after 4 iterations and 16 iterations:



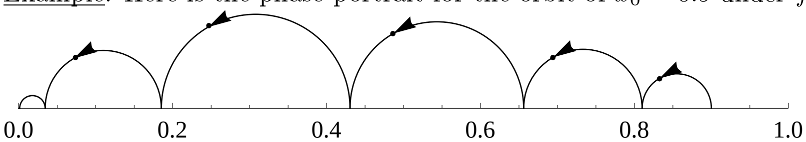
  - From the picture, we can see that the orbit spirals outward and approaches the 2-cycle 0, −1, 0, −1, ...

- Example: Plot the orbit of $x_0 = 0.4$ for the function $f(x) = 4x - 4x^2$.
  - Here are plots of the orbits after 4 iterations, 16 iterations, 100 iterations, and 500 iterations:

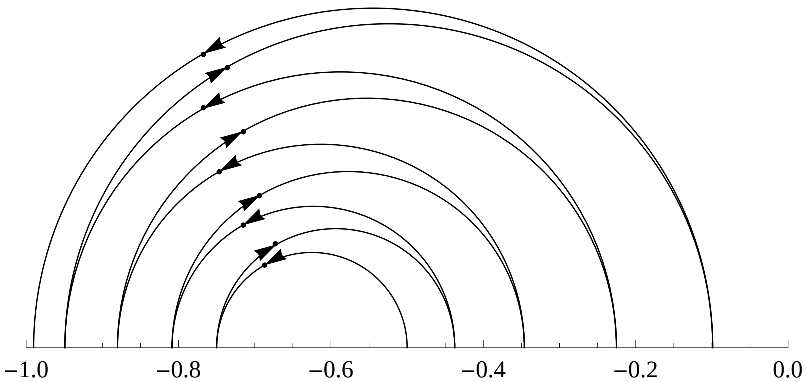


  - From the picture, we cannot really get any useful information about the orbit, except for the fact that it seems to meander unpredictably around the interval $[0, 1]$. (It certainly does not appear to be converging to anything obvious!)
  - In fact, this is an example of a chaotic orbit (the specifics of which we will analyze in a later chapter).

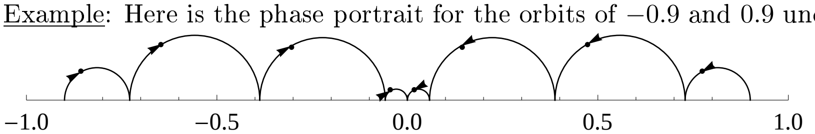
- Another graphical tool we can use to analyze orbits is the phase portrait: on a number line, we mark off the points in an orbit, and then draw arrows from one to the next.
  - Example: Here is the phase portrait for the orbit of $x_0 = 0.9$ under $f(x) = x^2$:



  - Example: Here is the phase portrait for the orbit of $x_0 = -0.5$ under $f(x) = x^2 - 1$:



- By combining phase portraits for several orbits, we can see some of the behaviors of the system.
  - Example: Here is the phase portrait for the orbits of -0.9 and 0.9 under $f(x) = x^3$:



  - The portrait suggests that the other orbits lying in $(-0.9, 0.9)$ are going to tend toward the fixed point $x = 0$.
- Although they can be useful for getting intuition about long-term qualitative behavior of orbits, we cannot really use these pictures (suggestive though they may be) to prove very much about the behavior of the function in question. To prove anything, we need some stronger tools.

1.3.2 Attracting and Repelling Fixed Points

- We will begin by studying fixed points. As we have seen in the examples, some systems have orbits which tend closer and closer to a fixed point (such as the map $f(x) = \sqrt{x}$ and the fixed point $x_0 = 1$), while other systems have orbits which move away from certain fixed points (such as the map $f(x) = x^2$ and the fixed point $x_0 = 1$).
  - We would like to explain why some fixed points "attract" nearby orbits while others "repel" them.
  - So suppose x_0 is a fixed point of f, and x is a nearby point.
  - Then $f(x)$ will be closer to x_0 than x is if $|f(x) - x_0| < |x - x_0|$.
  - Since $x_0 = f(x_0)$, we can equivalently write this as $\left| \dfrac{f(x) - f(x_0)}{x - x_0} \right| < 1$.
  - As x approaches x_0, notice that this quantity is the absolute value of the derivative $f'(x_0)$, assuming that f is differentiable.
  - Thus, we see that the behavior of a fixed point is closely linked with the value of $f'(x_0)$.
- Definition: If x_0 is a fixed point of the differentiable function f, we say x_0 is an attracting fixed point if $|f'(x_0)| < 1$, we say x_0 is a repelling fixed point if $|f'(x_0)| > 1$, and we say x_0 is a neutral fixed point if $|f'(x_0)| = 1$.

- The key result is that an attracting fixed point will attract nearby orbits, and a repelling fixed point will repel nearby orbits.
- If a fixed point x_0 has $f'(x_0) = 0$, we sometimes call it a superattracting fixed point, because orbits will approach it more quickly than a mere attracting fixed point.

- Theorem (Attracting Points): If x_0 is an attracting fixed point of the continuously differentiable function f, then there exists an open interval I containing x_0 such that, for any $x \in I$, $f^n(x) \in I$ for all $n \geq 1$. Furthermore, for any $x \in I$, it is true that $f^n(x) \to x_0$ as $n \to \infty$, and the convergence is exponentially fast. In fact, we can take I to be any interval containing x_0 with the property that there exists a constant $\lambda < 1$ such that $|f'(x)| < \lambda$ for all $x \in I$.
  - Remark: The speed of the convergence will depend on the value of $|f'(x_0)|$. From the argument at the end of the proof below, we see that the smaller this value is, the faster the orbits will converge to x_0. If $f'(x_0)$ happens to be equal to zero, then the convergence can be faster than exponential (hence the term "superattracting" fixed point).
  - Proof: Recall the statement of the Mean Value Theorem: if f is differentiable on the interval $[a, b]$, then there exists a value $c \in [a, b]$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
  - Now, by hypothesis, $|f'(x)|$ is continuous and $|f'(x_0)| < 1$. Thus, by standard properties of continuous functions, there exists a constant $\lambda < 1$ and an open interval I centered at x_0 such that $|f'(x)| < \lambda$ for all $x \in I$.
  - Now for any $x \in I$, apply the Mean Value Theorem to f on the interval whose endpoints are x_0 and x: then there exists a value c between x_0 and x for which $\dfrac{f(x) - f(x_0)}{x - x_0} = f'(c)$.
  - Taking the absolute value gives $\left|\dfrac{f(x) - x_0}{x - x_0}\right| = |f'(c)| < \lambda$, so that $|f(x) - x_0| < \lambda |x - x_0|$.
  - Since $\lambda < 1$ this implies $f(x) \in I$. Now applying the result for $f(x) \in I$ gives $f^2(x) \in I$, and a trivial induction gives $f^n(x) \in I$ for all $n \geq 1$.
  - Furthermore, we also have $\left|f^2(x) - x_0\right| < \lambda |f(x) - x_0| < \lambda^2 |x - x_0|$, and again by a trivial induction we see that $|f^n(x) - x_0| < \lambda^n |x - x_0|$ for all $n \geq 1$. Since $\lambda < 1$, as $n \to \infty$ the right-hand term goes to zero, so $f^n(x) \to x_0$, and the convergence is exponentially fast.

- We also have an analogous result for repelling points:

- Theorem (Repelling Points): If x_0 is a repelling fixed point of the continuously differentiable function f, then there exists an open interval I containing x_0 such that, for any $x \in I$ with $x \neq x_0$, there exists a positive integer n such that $f^n(x) \notin I$. In fact, we can take I to be any interval I with the property that there exists a constant $\lambda > 1$ such that $|f'(x)| > \lambda$ for all $x \in I$.
  - Note that we cannot expect to say anything about the limit of $f^n(x)$ as $n \to \infty$ (unlike in the case of an attracting point) because there is nothing to prevent a repelling point from being sent back into the "repelling interval" I once it escapes.
  - Proof: By the same argument as for the theorem on attracting points, there exists a constant $\lambda > 1$ and a finite open interval I centered at x_0 such that $|f'(x)| > \lambda$ for all $x \in I$.
  - By the Mean Value Theorem, we can again conclude that $|f(x) - x_0| > \lambda |x - x_0|$, and then by a trivial induction we see that $|f^n(x) - x_0| > \lambda^n |x - x_0|$, assuming that $f^{n-1}(x)$ lies in I.
  - If the orbit of x never left I, then we would have $|f^n(x) - x_0| > \lambda^n |x - x_0|$, but since $\lambda > 1$, as $n \to \infty$ the right-hand side tends to infinity. But $f^n(x)$ is assumed to lie in I for all n, meaning that I is an infinite interval: contradiction.

- Example: Find and classify the fixed points of $f(x) = x^3$ as attracting, repelling, or neutral.
  - It is easy to solve $x^3 = x$ to see that the fixed points are $x = 0$, $x = 1$, and $x = -1$.
  - Since $f'(x) = 3x^2$, we see that $\boxed{x = 0 \text{ is attracting}}$ and $\boxed{x = \pm 1 \text{ are repelling}}$.

- We can see the attracting and repelling nature of the fixed points by computing a few orbits.
- For example, the orbit of 0.9 is 0.9, 0.729, 0.387, 0.058, 0.0002, ..., while the orbit of 1.1 is 1.1, 1.331, 2.358, 13.110, 2253, ...
- Similarly, the orbit of -0.9 is -0.9, -0.729, -0.387, -0.058, ... and the orbit of -1.1 is -1.1, -1.331, -2.358, -13.110, -2253, ...

- Example: For each positive value of λ, find and classify the fixed points of the logistic map $p_\lambda(x) = \lambda x(1-x)$ as attracting, repelling, or neutral.
  - Setting $\lambda x(1-x) = x$ and solving yields $x = 0$ and $x = 1 - \dfrac{1}{\lambda}$.
  - We also have $p'_\lambda(x) = \lambda - 2\lambda x$, so $p'_\lambda(0) = \lambda$ and $p'_\lambda\left(1 - \dfrac{1}{\lambda}\right) = 2 - \lambda$.
  - So, the point $x = 0$ is attracting for $0 < \lambda < 1$, becomes neutral (and coincides with the other fixed point) for $\lambda = 1$, and is repelling for $\lambda > 1$.
  - Similarly, we see that $x = 1 - \dfrac{1}{\lambda}$ is repelling for $0 < \lambda = 1$, becomes neutral (and coincides with $x = 0$) for $\lambda = 1$, is $\boxed{\text{attracting for } 1 < \lambda < 3}$, is $\boxed{\text{neutral for } \lambda = 3}$, and is $\boxed{\text{repelling for } \lambda > 3}$.

1.3.3 Attracting and Repelling Cycles

- We can extend our definitions of attracting and repelling behavior to periodic cycles: since a periodic point of period n for f is the same as a fixed point of f^n, there is a natural way to extend the definition:
- Definition: We say that a periodic point x_0 for f is attracting (respectively, repelling or neutral) if x_0 is an attracting (respectively, repelling or neutral) fixed point for f^n.
  - A natural and immediate question is: can it happen that some points on a cycle are attracting and others are repelling?
  - In fact, this cannot occur for attracting points: if x_0 is an attracting fixed point for f^n, then the sequence $f^{kn}(x_0)$ will have limit x_0 as $k \to \infty$. Since a continuous function has the property that $a_i \to L$ implies $f(a_i) \to f(L)$, applying this fact to f and the sequence with $a_i = f^{ni}(x_0)$ shows that $f^{kn+1}(x_0)$ will have limit x_1. Repeating this argument shows that all of the other points in the cycle will attract nearby orbits.
  - However, the above argument cannot be easily adapted for repelling points.
- Using the chain rule, we can easily compute whether a periodic point is attracting, repelling, or neutral:
- Proposition (Attracting and Repelling Cycles): If $x_0, x_1, \dots, x_{n-1}, x_n = x_0$ is an n-cycle for f, then $\dfrac{d}{dx}[f^n(x)]$ at $x = x_i$ for any i is equal to $f'(x_{n-1}) \cdot f'(x_{n-2}) \cdots f'(x_1) \cdot f'(x_0)$. In particular, the points in the n-cycle are either all attracting, all repelling, or all neutral.
  - Proof: Let $g(x) = f^n(x)$. By an easy chain rule computation, $g'(x) = f'(f^{n-1}(x)) \cdot f'(f^{n-2}(x)) \cdot \cdots \cdot f'(f(x)) \cdot f'(x)$. Setting $x = x_0$ yields $g'(x_0) = f'(x_{n-1}) \cdot f'(x_{n-2}) \cdots f'(x_1) \cdot f'(x_0)$: in other words, the value $g'(x_0)$ is equal to the product of f' evaluated at each of the points in the cycle.
  - Applying this result for each point in the n-cycle shows that $g'(x_0) = g'(x_1) = \cdots = g'(x_{n-1})$, so, by our criteria for attracting and repelling points, this means all the points on the cycle are either all attracting, all repelling, or all neutral.
- Example: Show that the 2-cycle $\{0, -1\}$ for the function $f(x) = x^2 - 1$ is attracting.
  - We have $f'(x) = 2x$, so we need to compute $f'(0)f'(-1) = 0$.
  - This has absolute value less than 1, so the 2-cycle is $\boxed{\text{attracting}}$.

- Example: Show that every periodic cycle lying in $(0,1)$ for the doubling function $D(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/2 \\ 2x-1 & \text{if } 1/2 \le x < 1 \end{cases}$ is repelling.
  - Observe that $D'(x) = 2$ for every $x \in (0,1)$ except for $x = 1/2$ (where the derivative is undefined, due to the discontinuity). Notice also that $1/2$ does not lie in a periodic cycle, so we can safely ignore it.
  - Thus, on any n-cycle, the value of the derivative of D^n will be 2^n. Since this has absolute value larger than 1, $\boxed{\text{every } n\text{-cycle is repelling}}$.
  - From this we can see another way to understand the "chaotic" behavior of the function: every rational number with odd denominator lies on a repelling n-cycle, so as we iterate the function, any two nearby points will be pushed away from one another.

- Example: Classify the periodic cycle containing 0 for the function $f(x) = 1 - \frac{x}{3} + 2x^2 - \frac{2x^3}{3}$ as attracting, repelling, or neutral.
  - The orbit of 0 is 0, 1, 2, 3, 0, ... , so it is a 4-cycle.
  - We have $f'(x) = -\frac{1}{3} + 4x - \frac{4}{3}x^2$, so $f'(0) = -\frac{1}{3}$, $f'(1) = \frac{5}{3}$, $f'(2) = -\frac{1}{3}$, and $f'(3) = -\frac{19}{3}$.
  - Thus, if $g = f^4$, we have $g' = \left(-\frac{1}{3}\right)\left(\frac{5}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{19}{3}\right) = -\frac{85}{81}$ at each point on the 4-cycle, so the cycle is $\boxed{\text{repelling}}$.

- Example: Classify the periodic cycle containing $-\frac{1}{3}$ for the function $f(x) = x^2 - \frac{7}{9}$ as attracting, repelling, or neutral.
  - The orbit of 1 is $\left\{-\frac{1}{3}, \frac{2}{3}\right\}$, which is a 2-cycle.
  - We have $f'(x) = 2x$, so $f'(-\frac{1}{3}) = -\frac{2}{3}$ and $f'(\frac{2}{3}) = \frac{4}{3}$.
  - Thus, if $g = f^2$, we have $g' = -\frac{8}{9}$ at both points on the 2-cycle, so the cycle is $\boxed{\text{attracting}}$.

- Example: Classify the periodic cycle containing 1 for the function $f(x) = \sqrt{3} - \frac{1}{x}$ as attracting, repelling, or neutral.
  - The orbit of 1 is $\left\{1, \sqrt{3}-1, \frac{1}{\sqrt{3}+1}, -1, \sqrt{3}+1, \frac{1}{\sqrt{3}-1}\right\}$, which is a 6-cycle.
  - We have $f'(x) = \frac{1}{x^2}$, so $f'(\pm 1) = 1$, $f'(\sqrt{3} \pm 1) = \frac{1}{\left(\sqrt{3}\pm 1\right)^2}$, and $f\left(\frac{1}{\sqrt{3}\pm 1}\right) = \left(\sqrt{3}\pm 1\right)^2$, where the choices of $\pm$ correspond in each case.
  - Thus, if $g = f^6$, we have $g' = 1$ at each point on the 6-cycle, so the cycle is $\boxed{\text{neutral}}$.

- Example: For $3 < \lambda \le 4$, determine (in terms of λ) when the 2-cycle of the the logistic map $p_\lambda(x) = \lambda x(1-x)$ is attracting, neutral, or repelling.
  - We computed earlier that the points on the 2-cycle are the two roots of the quadratic $q(x) = \lambda^2 x^2 - (\lambda + \lambda^2)x + (1+\lambda)$, which (explicitly) are $r_1, r_2 = \frac{1 + \lambda \pm \lambda\sqrt{\lambda^2 - 2\lambda - 3}}{2\lambda}$, and that they are real-valued on the given range for λ.
  - To determine the behavior of the 2-cycle, we need to compute $p_\lambda'(r_1) \cdot p_\lambda'(r_2) = 4\lambda^2(\frac{1}{2} - r_1)(\frac{1}{2} - r_2)$.
  - One can compute this by slogging it out, but a slicker way is to observe that $\lambda^2(x - r_1)(x - r_2) = q(x)$, so, upon setting $x = \frac{1}{2}$, we obtain $\frac{\lambda^2}{4} - \frac{1}{2}(\lambda + \lambda^2) + (1+\lambda) = \lambda^2(\frac{1}{2} - r_1)(\frac{1}{2} - r_2)$. Multiplying through by 4 gives $p_\lambda'(r_1) \cdot p_\lambda'(r_2) = -\lambda^2 + 2\lambda + 4$.

- On the interval $(3,4]$, this quadratic takes values in $(-1,1)$ for $3 < \lambda < 1+\sqrt{6}$, is equal to -1 at $1+\sqrt{6}$, and is less than -1 for $1+\sqrt{6} < \lambda \le 4$.
- Thus, we conclude that the 2-cycle is $\boxed{\text{attracting for } 3 < \lambda < 1+\sqrt{6}}$, is $\boxed{\text{neutral when } \lambda = 1+\sqrt{6}}$, and is $\boxed{\text{repelling when } 1+\sqrt{6} < \lambda \le 4}$.

1.3.4 Weakly Attracting and Weakly Repelling Points (and Cycles)

- We have determined the behaviors of attracting and repelling points and cycles. Let us now turn our attention to studying orbits near neutral fixed points and cycles, after examining a few examples.
- Example: Examine the orbits near the neutral fixed point $x_0 = 0$ of $f(x) = x + x^2$.
  - The first ten terms in the orbit of 0.1 under f are 0.1, 0.11, 0.1221, 0.1370, 0.1558, 0.1800, 0.2125, 0.3240, 0.4289, 0.6129.
  - Similarly, the first ten terms in the orbit of -0.1 under f are -0.1, -0.09, -0.0819, -0.075, -0.070, -0.0605, -0.0569, -0.0536, -0.0507, -0.0482.
  - We can see that the orbits with small positive x are repelled (slowly) from 0, while the orbits with small negative x are attracted (slowly) from 0.
- Example: Examine the orbits near the neutral fixed point $x_0 = 0$ of $g(x) = x + x^3$.
  - For f, the first ten terms in the orbit of 0.1 is 0.1, 0.101, 0.1020, 0.1031, 0.1042, 0.1053, 0.1065, 0.1077, 0.1089, 0.1102.
  - Similarly, the first ten terms in the orbit of -0.1 under f are -0.1, -0.101, -0.1020, -0.1031, -0.1042, -0.1053, -0.1065, -0.1077, -0.1089, -0.1102.
  - We can see that the orbits with near 0 seem to be repelled (quite slowly) from 0.
- Example: Examine the orbits near the neutral fixed point $x_0 = 0$ of $h(x) = x - x^3$.
  - For f, the first ten terms in the orbit of 0.1 is 0.1, 0.099, 0.0980, 0.0971, 0.0962, 0.0953, 0.0944, 0.0936, 0.0928, 0.0920.
  - Similarly, the first ten terms in the orbit of -0.1 under f are -0.1, -0.099, -0.0980, -0.0971, -0.0962, -0.0953, -0.0944, -0.0936, -0.0928, -0.0920.
  - We can see that the orbits with near 0 seem to be attracted (quite slowly) to 0.
- We define the notion of weakly attracting / weakly repelling fixed point based on the behavior of nearby orbits:
- Definition: If x_0 is a neutral fixed point of the differentiable function f, we say x_0 is weakly attracting if there exists an open interval I containing x_0 such that for any $x \in I$, $f^n(x) \to x_0$ as $n \to \infty$. We also say a neutral periodic point for f of period n is weakly attracting if it is a weakly attracting fixed point for f^n.
  - In other words, a weakly attracting fixed point (or cycle) is one that attracts nearby orbits. All of our results for attracting fixed points (and cycles) that only invoke the "attracting orbit" property will also hold for weakly attracting fixed points (and cycles).
  - There is also a "one-sided" version of weak attraction that can occur when $f'(x_0) = 1$: a fixed point is weakly attracting on the left if there is an $\epsilon > 0$ such that every point $x \in (x_0 - \epsilon, x_0)$ has $f^n(x) \to x_0$. (In other words, if it attracts orbits on its left.)
  - Similarly, we say a point is weakly attracting on the right if there is an $\epsilon > 0$ such that every point $x \in (x_0, x_0 + \epsilon)$ has $f^n(x) \to x_0$. (In other words, if it attracts orbits on its right.)
- Definition: If x_0 is a neutral fixed point of the differentiable function f, we say x_0 is weakly repelling if there exists an open interval I containing x_0 such that for any $x \in I$ (except $x = x_0$), there exists an n such that $f^n(x) \notin I$. We say a neutral periodic point for f of period n is weakly repelling if it is a weakly repelling fixed point for f^n.

- In other words, a weakly repelling fixed point is one that repels nearby orbits.
- Like with weakly attracting points, there are also "one-sided" versions of weak repulsion (which, likewise, only occurs when $f'(x_0) = 1$): we say x_0 is weakly repelling on the left if there exists an $\epsilon > 0$ such that for every $x \in (x_0 - \epsilon, x_0)$, there exists an n such that $f^n(x) \notin (x_0 - \epsilon, x_0]$, and similarly we say x_0 is weakly repelling on the right if there exists an $\epsilon > 0$ such that for every $x \in (x_0, x_0 + \epsilon)$, there exists an n such that $f^n(x) \notin [x_0, x_0 + \epsilon)$.
- We would like to determine when a neutral fixed point is weakly attracting or repelling on each side.

- Theorem (Neutral Points): Suppose x_0 is a neutral fixed point for a function f with $f'(x_0) = 1$. Furthermore, assume that there is an integer $k \geq 2$ such that (i) the $(k+1)$st derivative of f is continuous at x_0, (ii) the value $f^{(k)}(x_0) \neq 0$, and (iii) that $f^{(d)}(x_0) = 0$ for all $1 < d < k$. If k is odd, the point x_0 is weakly attracting if $f^{(k)}(x_0) < 0$ and it is weakly repelling if $f^{(k)}(x_0) > 0$. If k is even, the point x_0 is weakly repelling on the left and weakly attracting on the right if $f^{(k)}(x_0) < 0$, and it is weakly attracting on the left and weakly repelling on the right if $f^{(k)}(x_0) > 0$.
  - The statement requires some unpacking. Ultimately, it says that the behavior of a neutral fixed point is controlled by the order and the sign of the first nonzero derivative of f (beyond f') at that point.
  - The key ingredient in the proof is Taylor's theorem: we will find a polynomial approximation to $f(x)$ that is simple enough (but also accurate enough) for us to characterize the behavior of the orbits near x_0.
  - Proof: For clarity, we make the change of variables $y = x - x_0$, so that the fixed point is now at $y = 0$.
    * Taylor's Theorem says: if $f(y)$ is a function whose $(k+1)$st derivative is continuous, and $T_k(x)$ is the kth Taylor polynomial $T_k(y) = \sum_{d=0}^{k} \frac{f^{(d)}(0)}{d!} y^d$ for $f(y)$ at $y = 0$, then $|f(y) - T_k(y)| \leq M \cdot \frac{|y|^{k+1}}{(k+1)!}$, where M is any constant such that $\left|f^{(k+1)}(t)\right| \leq M$ for all t in the interval $[-|y|, |y|]$.
  - In our case, everything except the 0th, 1st, and kth terms of the Taylor polynomial are zero, and we get the simple expression $T_k(y) = y + \frac{f^{(k)}(0)}{k!} y^k$.
    * Then, if we take a small enough interval around zero, we can arrange it so that the error term $|f(x) - T_k(x)|$ is less than $\frac{1}{2} \left| \frac{f^{(k)}(0)}{k!} y^k \right|$.
    * Explicitly: since $f^{(k+1)}$ is continuous, there is an open interval I around 0 and some M such that $\left|f^{(k+1)}(t)\right| \leq M$ on I. Then the subinterval of I where $|y| < \frac{k+1}{2M} \left|f^{(k)}(0)\right|$ has the desired property.
    * So, for all y in this interval, we can conclude that $f(y)$ always lies between $y + \frac{1}{2} \cdot \frac{f^{(k)}(0)}{k!} y^k$ and $y + \frac{3}{2} \cdot \frac{f^{(k)}(y)}{k!} y^k$.
  - In particular, for small enough $|y|$, $f(y)$ lies on the same side of 0 as y does, and $f(y) - y$ has the same sign as $f^{(k)}(0) \cdot y^k$. Thus, we just need to determine whether $f(y)$ is closer or farther from 0 than y is, which is to say, whether $f(y) - y$ has the same or opposite sign as y, respectively.
    * If k is even, then $f^{(k)}(0) \cdot y^k$ has the same sign as $f^{(k)}(0)$, so 0 is weakly repelling on the left and weakly attracting on the right if $f^{(k)}(0) < 0$, and weakly attracting on the left and weakly repelling on the right if $f^{(k)}(0) > 0$.
    * If k is odd and $f^{(k)}(0) > 0$, then $f(y) - y$ has the same sign as y so 0 is weakly repelling. If $f^{(k)}(0) < 0$, then $f(y) - y$ has the opposite sign as y, meaning that 0 is weakly attracting.

- Example: Classify the neutral fixed point $x_0 = 0$ for $a(x) = x + x^2$, $b(x) = x - x^2$, $c(x) = x + x^3$, and $d(x) = x - x^3$ as weakly attracting or weakly repelling for orbits on each side.
  - For a, we have $a'(0) = 1$ and $a''(0) = 2$, so $k = 2$ and then x_0 is $\boxed{\text{weakly attracting on the left}}$ and $\boxed{\text{weakly repelling on the right}}$.

- For b, we have $b'(0) = 1$ and $b''(0) = -2$, so $k = 2$ and then x_0 is $\boxed{\text{weakly repelling on the left}}$ and $\boxed{\text{weakly attracting on the right}}$.
- For c, we have $c'(0) = 1$, $c''(0) = 0$, and $c'''(0) = 6$, so $k = 3$ and then x_0 is $\boxed{\text{weakly repelling}}$ on both sides.
- For d, we have $d'(0) = 1$, $d''(0) = 0$, and $d'''(0) = -6$, so $k = 3$ and then x_0 is $\boxed{\text{weakly attracting}}$ on both sides.

- Example: Classify the neutral fixed point of $f(x) = \tan^{-1}(x)$ as weakly attracting or weakly repelling for orbits on each side.
  - Notice that $f(0) = 0$ and $f'(x) = \dfrac{1}{1+x^2}$, so the only neutral fixed point is $x = 0$. (In fact it is the only fixed point.)
  - Since $f''(0) = 0$ and $f'''(0) = -2$, we see that $k = 3$.
  - By the classification, we see that $\boxed{\text{0 is weakly attracting}}$.

- Notice that the theorem above does not treat all possible neutral fixed points: we did not treat the case where $f'(x_0) = 1$ but $f^{(k)}(x_0) = 0$ for all $k \geq 2$, nor did we treat the case where x_0 is a neutral fixed point with $f'(x_0) = -1$.
  - If we have a neutral fixed point with $f'(x_0) = 1$ but $f^{(k)}(x_0) = 0$ for all $k \geq 2$, then the Taylor series of f will just be $T(x) = x$, and so it will provide no useful information: some other kind of technique would be needed to study the behavior of the orbits.
  - Fortunately, aside from $f(x) = x$, whose orbits are obvious, it is rare to encounter such functions.
  - For completeness, a standard example is $f(x) = \begin{cases} x + e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$, which has $f(0) = 0$, $f'(0) = 1$, and $f^{(n)}(0) = 0$ for all $n \geq 2$. It is not completely obvious that the higher derivatives of f even exist, but they can be computed using some careful limit computations.

- If we are given a neutral fixed point with $f'(x_0) = -1$, notice that we can apply the theorem to analyze the behavior of x_0 as a fixed point of $g = f^2$, because we have $g'(x_0) = f'(x_0)f'(x_0) = 1$ by the chain rule.
  - Ultimately, these neutral fixed points carry the additional complication that a point on one side of x_0 will flip to the other side after applying f. In some cases it is easy to see that both sides are attracting or repelling, so the "flipping" does not make a difference.
  - However, it can happen (e.g., with the function $f(x) = -x + x^2$ below) that one side will move points closer to x_0, and the other side will move points farther away from x_0. To decide which behavior wins out, it is necessary to study x_0 as a fixed point of f^2.
  - Explicitly: if x_0 is weakly attracting as a fixed point of f^2, then it is weakly attracting as a fixed point of f, and similarly, if x_0 is weakly repelling as a fixed point of f^2, then it is weakly repelling as a fixed point of f^2.

- Example: Classify the neutral fixed point $x_0 = 0$ of $f(x) = -x + x^2$ as weakly attracting or weakly repelling for orbits on each side.
  - Observe first that that if x is small and positive, then $|f(x)| = x - x^2$, so f moves positive points closer to zero. However, if x is small and negative, then $|f(x)| = |x| + x^2$, so f moves negative points farther away from zero. Since f maps small positive numbers to small negative numbers (and vice versa), it is not clear whether the "attracting" behavior or the "repelling" behavior will win the race (so to speak) as we continue iterating f.
  - Notice that $f(0) = 0$ and $f'(0) = -1$, so to classify the orbit behavior we should look at 0 as a fixed point of $g(x) = f^2(x) = x - 2x^3 + x^4$.
  - We have $g(0) = 0$, $g'(0) = 1$, $g''(0) = 0$, and $g'''(0) = -2$, so $k = 3$ and thus 0 is weakly attracting as a fixed point of g.

- Thus, $\boxed{\text{0 is weakly attracting}}$ for f as well.

- Example: Classify the neutral fixed point $x_0 = 0$ of $f(x) = -x + x^2 - x^3$ as weakly attracting or weakly repelling for orbits on each side.
  - Notice that $f(0) = 0$ and $f'(0) = -1$, so to classify the orbit behavior we need to look at 0 as a fixed point of $g(x) = f^2(x) = x + 4x^5 - 6x^6 + 6x^7 - 3x^8 + x^9$.
  - We have $g(0) = 0$, $g'(0) = 1$, $g^{(2)}(0) = g^{(3)}(0) = g^{(4)}(0) = 0$, and $g^{(5)}(0) = 480$, so $k = 5$ and thus 0 is weakly repelling as a fixed point of g.
  - Thus, $\boxed{\text{0 is weakly repelling}}$ for f as well.
  - Notice that this example only differs from the previous one in the degree-3 term of f. In particular, we can see that knowing the first two nonzero terms of the Taylor series for f is not enough to determine the orbit behavior of x_0 as attracting or repelling: later terms can also affect the result.

- We can also use the theorem to classify the behavior of orbits near a neutral periodic point x_0: the starting point is to determine the orbit behavior for x_0 as a neutral fixed point of f^n.
  - In the event that x_0 is weakly attracting (or weakly repelling) for f^n, essentially by definition we can conclude that x_0 is a weakly attracting (or weakly repelling) periodic point for f.
  - However, if x_0 is weakly attracting in one direction and weakly repelling in the other direction as a fixed point of f^n, the behavior of the periodic cycle of f containing x_0 is trickier:
    - If the derivative of f^n at x_0 is $+1$, then cycles starting on one side of x_0 will be attracting and cycles on the other side will be repelling.
    - If the derivative of f^n at x_0 is -1, then a point on the "attracting" side of x_0 will flip to the "repelling" side after applying f^n (and vice versa), so to decide which behavior wins out, it is necessary to study x_0 as a fixed point of f^{2n}.

- Example: Show that 0 lies on a neutral 2-cycle for $p(x) = 1 + x - 6x^2 + 4x^3$, and classify the behavior near 0 as weakly attracting or repelling on each side.
  - We have $p(0) = 1$, $p(1) = 0$, and also $p'(x) = 1 - 12x + 12x^2$ so $p'(0) = p'(1) = 1$. Thus, the 2-cycle $\{0, 1\}$ is neutral.
  - We can expand (ideally with a computer) to find $p(p(x)) = x - 64x^3 + 192x^4 + 192x^5 - 1344x^6 + 1920x^7 - 1152x^8 + 256x^9$.
  - Thus, by the neutral point classification (here, $k = 3$, the first derivative is 1, and the third derivative is negative) we see that the cycle is $\boxed{\text{weakly attracting}}$.

- Example: Show that $q(x) = x^2 - \frac{5}{4}$ has a neutral 2-cycle, and classify the behavior near 0 as weakly attracting or repelling on each side.
  - We have $\dfrac{q(q(x)) - x}{q(x) - x} = x^2 + x - \dfrac{1}{4}$, whose roots are $r_1, r_2 = \dfrac{-1 \pm \sqrt{2}}{2}$.
  - Note that $q'(x) = 2x$, so $q'(r_1) = -1 + \sqrt{2}$ and $q'(r_2) = -1 - \sqrt{2}$, so since $(-1 + \sqrt{2})(-1 - \sqrt{2}) = -1$, the cycle is indeed neutral.
  - To analyze the attracting behavior, we look at the behavior of r_1 as a fixed point of $g(x) = q^2(x) = x^4 - \frac{5}{2}x^2 + \frac{5}{16}$.
  - We have $g(r_1) = r_1$, $g'(r_1) = -1$, and $g''(r_1) = 4 - 6\sqrt{2}$. Thus, we have $k = 2$, and so the cycle is neither weakly attracting nor weakly repelling: as a fixed point of g, we can check that r_1 weakly attracting on the left and weakly repelling on the right.
  - To study the nearby orbits, we must look at $h(x) = q^4(x)$. Using a computer, we can evaluate $h(r_1) = r_1$, $h'(r_1) = 1$, $h''(r_1) = 0$, and $h'''(r_1) = 120(\sqrt{2} - 2)$.

- Thus, by the neutral point classification (here, $k = 3$, the first derivative is 1, and the third derivative is negative), we see that r_1 is a weakly attracting fixed point of q^4: thus, we conclude that the 2-cycle $\{r_1, r_2\}$ for f is $\boxed{\text{weakly attracting}}$.
- Using a computer, we can compute that $r_1 = 0.207107$, $r_2 = -1.207107$, and that the orbit of 0.2 (to six decimal places) is 0.2, -1.21, 0.2141, -1.204162, 0.200004, -1.209998, -0.214096, -1.204163, 0.200008, ...
- We can see that, after every four repetitions, the orbit inches closer to the 2-cycle (as the above analysis dictates it will) but the convergence is exceedingly slow!

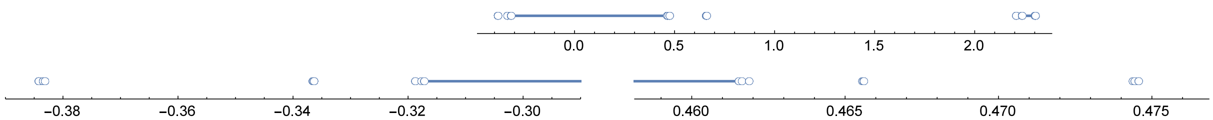
- A natural question is: how fast does a weakly attracting fixed point attract nearby orbits?
  - For simplicity, let us suppose that $f(x) = x - cx^k$ for some positive constant c and some $k \geq 2$, and study the orbits of small positive x.
  - Equivalently, we want to estimate how fast the sequence $x_{n+1} = x_n - cx_n^k$ approaches zero, for a given x_0.
  - If we rewrite the definition as $x_{n+1} - x_n = -cx_n^k$, then because the sequence is nearly constant, we can approximate this "difference equation" with the differential equation $\dfrac{dx}{dt} = -cx^k$, with initial condition $x = x_0$.
  - This is an easy separable equation whose solution has the form $x(t) = (Ct + D)^{-1/(k-1)}$ for constants C and D in terms of x_0, k, and c. (One can compute the constants, but we are only interested in the rough behavior.)
  - The solution to the difference equation is then approximately $x_n \approx (Cn + D)^{-1/(k-1)}$. As $n \to \infty$, this does tend to zero as we claimed, but it does so rather slowly: for $k = 2$, it goes to zero like n^{-1}, and for $k = 3$, it goes to zero like $n^{-1/2}$. This is very slow compared to the exponential convergence λ^{-n} for some $\lambda < 1$ possessed by attracting fixed points.
  - We will remark that a change of variables combined with a Taylor's theorem argument much like the one in the classification proof will allow us to extend this analysis extends to all weakly attracting fixed points. (We will not bother with the details.)

1.3.5 Basins of Attraction

- Our theorems on attracting fixed points and cycles are useful in describing the orbits of points "sufficiently close" to the attracting point or cycle, but they suffer from the limitation that they do not tell us explicitly what orbits will eventually fall towards them.
  - It is possible to get actual numeric bounds out of the proof of the theorem for attracting fixed points, namely: if x_0 is an attracting fixed point, then on the largest interval I containing x_0 with $|f'(x)| < 1$ for all $x \in I$, every orbit will approach x_0.
  - Example: For the function $f(x) = x^3$, clearly $x_0 = 0$ is an attracting fixed point since $f(0) = f'(0) = 0$. Since $f'(x) = 3x^2$, our result implies that every orbit that starts in the interval $\left(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$ will tend to 0 as we iterate f.
  - But this is not the strongest possible result: in fact, any orbit in the larger interval $(-1, 1)$ will tend to 0, since $f^n(x) = x^{3^n}$ clearly tends to 0 (quite rapidly!) for any such point.
  - One reason we do not get this larger interval is that, in the proof of the attracting fixed point theorem we gave, we actually wanted to analyze the function $\left|\dfrac{f(x) - x_0}{x - x_0}\right|$, rather than $|f'(x)|$. (For x near x_0, these two values are close together by the continuity of $f'(x)$, as we already saw.)

- Definition: If x_0 is an attracting (or weakly attracting) fixed point of f, the basin of attraction (or attracting basin) for x_0 is the set of all points x such that $f^n(x) \to x_0$ as $n \to \infty$. (In other words, it is the points whose orbits attract to x_0.) The immediate basin of attraction for x_0 is the largest interval around x_0 contained in the basin of attraction.

- In general, the structure of the basin of attraction can be quite complicated: it is frequently an infinite union of disjoint intervals.
- For example, here are a few plots (on different scales) of the attracting basin for the attracting fixed point $x_0 = 1$ of the function $f(x) = \dfrac{2x^2(1-5x+2x^2)}{(3-5x)(1+2x-x^2)}$:



- It is generally much easier to compute the immediate basin of attraction than the full basin (though of course, if f is sufficiently complicated, we can usually only compute an approximation).

- A starting point for computing the immediate basin of attraction is to find the set of points that f moves closer to x_0:

- Proposition: Suppose that x_0 is a (weakly) attracting fixed point of f and λ be any positive constant less than 1. If S is the set of points x such that $x = x_0$ or $\left|\dfrac{f(x)-x_0}{x-x_0}\right| < \lambda$, and I is the largest interval of the form $(x_0 - c, x_0 + c)$ lying in S, then I lies in the immediate basin of attraction for x_0 under f.

  - Proof: Let I be the interval defined above. By definition, if $x \in I$, then $|f(x) - x_0| < (1-\epsilon)\,|x - x_0|$: thus $f(x)$ is closer to x_0 than x is. Furthermore, because I is symmetric about x_0, we see that $f(x)$ also lies in I.
  - We can then apply the result repeatedly to see that (by a trivial induction) $|f^n(x) - x_0| < \lambda^n\,|x - x_0|$, so $f^n(x) \to x_0$ as $n \to \infty$, meaning x is in the basin of attraction for x_0. Since I is an open interval around x_0 and every point in it lies in the basin of attraction, I lies in the immediate basin.

- Example: Find the immediate basin of attraction for the attracting fixed point $x_0 = 0$ of $f(x) = x^3$.

  - We start by determining when $\left|\dfrac{f(x)-0}{x-0}\right| = \left|x^2\right|$ is less than 1. Clearly, this holds for $-1 < x < 1$, so the immediate basin contains the interval $(-1, 1)$.
  - Observe that the basin is bounded by the points $x = -1$ and $x = 1$, which are both fixed points for f, so the immediate basin does not contain them. So, the immediate basin is actually $\boxed{(-1,1)}$.
  - In fact, the interval $(-1, 1)$ is actually the entire basin of attraction for $x_0 = 0$, because any value in $(-\infty, -1)$ will have orbit tending to $-\infty$, and any value in $(1, \infty)$ will have orbit tending to ∞.

- Example: Find the immediate basin of attraction for the weakly attracting fixed point of $f(x) = x - x^3$.

  - The fixed point is $x = 0$, so we start by finding those $x \neq 0$ such that $\left|\dfrac{f(x)}{x}\right| < 1$: namely, such that $\left|1 - x^2\right| < 1$.
  - This relation is satisfied whenever $|x| < \sqrt{2}$ (except for $x = 0$, but it is certainly in the immediate basin) so the immediate basin of attraction contains $(-\sqrt{2}, \sqrt{2})$.
  - But now note that $f(\sqrt{2}) = -\sqrt{2}$ and $f(-\sqrt{2}) = \sqrt{2}$, so since these two points lie on a 2-cycle, neither of them is in the basin of attraction.
  - Thus, we conclude that the immediate basin is $\boxed{(-\sqrt{2}, \sqrt{2})}$.

- In each of the above examples, the endpoints of the immediate basin for the (weakly) attracting fixed point have been fixed points, or points lying in a 2-cycle. This is not a coincidence:

- Theorem (Immediate Attracting Basin): If x_0 is a (weakly) attracting fixed point of the continuous function f with immediate basin of attraction I, then I is an open interval of one of the following types: (i) $(-\infty, \infty)$, (ii) $(-\infty, a)$ or (a, ∞) for a a fixed point, (iii) (a, b) for both a and b fixed points or with one a fixed point and the other a preimage of it, or (iv) (a, b) where $\{a, b\}$ is a 2-cycle.

- Remark: Recall that we say x is a preimage (or inverse image) of y under the map f if $f(x) = y$.
- Proof: Note that I is always an interval containing x_0, and it is also open because if it contained an endpoint, continuity would allow us to extend the interval past the endpoint. Let $\overline{I}$ be the topological closure of I: namely, I along with any finite endpoints, so that (for example) we have $\overline{(a,b)} = [a,b]$.
- By continuity, $f(\overline{I})$ is contained in $\overline{I}$, since $f(I)$ is contained in I by the definition of the immediate basin. If a is a finite endpoint of $\overline{I}$ (assuming it has one), then $f(a)$ cannot be contained in I: otherwise the orbit of a would attract to x_0, contrary to the assumption that a is not in the attracting basin. Thus, $f(a)$ must also be a finite endpoint of $\overline{I}$.
- If $I = (-\infty, \infty)$ we are done. If $I = (a, \infty)$ or $(-\infty, a)$, then we must have $f(a) = a$ since a is the only finite endpoint of I.
- Now suppose $I = (a, b)$. Then $f(a)$ and $f(b)$ are each either a or b. If $f(a) = a$ and $f(b) = b$ they are both fixed points of f.
- If $f(a) = f(b) = a$ or $f(a) = f(b) = b$ one is a fixed point and the other is a preimage of it.
- Finally, if $f(a) = b$ and $f(b) = a$, then $\{a, b\}$ forms a 2-cycle. This exhausts all the possibilities, so we are done.

- Using the theorem, we can compute the immediate basin of attraction of any (weakly) attracting point x_0: we need only compute all the fixed points of f, their preimages, and the 2-cycles of f. Then the closest such points on each side of x_0 will be the endpoints of the immediate basin. (Or $-\infty$ or ∞, if there are no such points.)

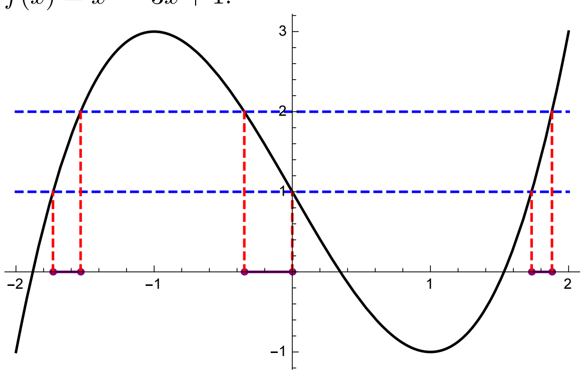
- Example: For $1 < \lambda < 3$, find the immediate basin of attraction inside $[0, 1]$ for the attracting fixed point of the logistic map $p_\lambda(x) = \lambda x(1 - x)$.
  - We computed earlier that the fixed point $x_0 = 1 - \frac{1}{\lambda}$ is attracting when $1 < \lambda < 3$, and we also showed that there is no real-valued 2-cycle for these values of λ.
  - We can also easily compute that the preimages of 0 are 0 and 1.
  - Thus, the possible endpoints of the immediate basin are $-\infty$, 0, 1, ∞. Since $x_0 = 1 - \frac{1}{\lambda}$ is between 0 and 1, the attracting basin must be $\boxed{(0, 1)}$, independent of λ.
  - Note that this proof is essentially nonconstructive: we do not know anything about how long it will take the orbit of any particular point in $(0, 1)$ to move close enough that it will be exponentially attracted to the fixed point. (All that we know is that it will eventually happen.)

- Example: Find the immediate basin of attraction for each attracting fixed point of $f(x) = -\frac{1}{2}x - \frac{5}{2}x^2 - x^3$.
  - Solving $f(x) = x$ produces $x = 0, -1, -\frac{3}{2}$. Since $f'(-1) = \frac{3}{2}$ it is repelling, but $f'(0) = -\frac{1}{2}$ and $f'(-\frac{3}{2}) = \frac{1}{4}$, so both 0 and $-\frac{3}{2}$ are attracting.
  - To compute the immediate basins, we will look for possible endpoints. Numerically solving the degree-6 polynomial $\frac{f(f(x)) - x}{f(x) - x} = 0$ yields one real-valued 2-cycle: $\{-2.4275, 0.7867\}$. We can also easily compute that the preimages of -1 are -1, $\frac{1}{2}$, and -2.
  - Thus, the possible endpoints for the immediate basins are $-\infty$, -2.4275, -2, -1, 0.5, 0.7867, ∞.
  - Since 0 lies in $(-1, \frac{1}{2})$, the immediate basin of 0 must be $\boxed{(-1, \frac{1}{2})}$. Similarly, since $-\frac{3}{2}$ lies in $(-2, -1)$, the immediate basin of $-\frac{3}{2}$ is $\boxed{(-2, -1)}$.

- Assuming we can compute an open interval lying in the immediate basin of attraction for a fixed point, we can give a description of the entire basin of attraction:

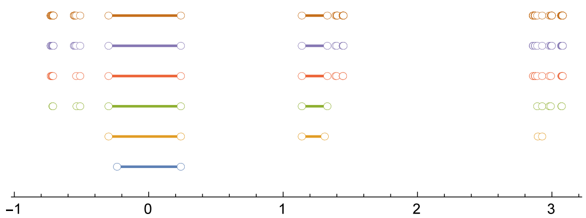
- Proposition: If x_0 is a (weakly) attracting fixed point of the continuous function f and I is any open interval containing x_0 that lies in the immediate basin of attraction, then the full basin of attraction B_{x_0} is given by $B_{x_0} = \bigcup_{n=0}^{\infty} f^{-n}(I) = I \cup f^{-1}(I) \cup f^{-2}(I) \cup \cdots$.
  - Recall that if S is a set, then $f^{-1}(S) = \{x : f(x) \in S\}$ is the inverse image (or preimage) of S under f, the set of all points which f maps into S. We then take $f^{-n}(S)$ to be the nth iterate of the preimage operation, or, equivalently, $f^{-n}(S) = \{x : f^n(x) \in S\}$.
  - Proof: Suppose x is in the basin of attraction of x_0. Then $f^n(x) \to x_0$ so by definition, for sufficiently large n we must have $f^n(x) \in I$: but this is immediately equivalent to $x \in f^{-n}(I)$. Conversely, if $f^n(x) \in I$, then since I is in the immediate basin of attraction we see that $f^k(f^n(x)) \to x_0$ as $k \to \infty$, and this is equivalent to saying that $f^k(x) \to x_0$ as $k \to \infty$.

- What the previous proposition says is: we can compute the full basin of attraction simply by finding an interval I that lies in the immediate basin, computing the sequence of preimages $f^{-n}(I)$ as $n \to \infty$, and taking the union.
  - In fact, each preimage will contain the previous one because $f(I) \subseteq I$, so taking the union is (vaguely) superfluous.
  - Computing preimages rapidly becomes intractable to do exactly (even for polynomials of small degree), and the iterated inverse image can become very complicated. As a theoretical tool, the proposition is therefore somewhat limited.
  - Computationally, however, the proposition is quite useful: if f is continuous on I, then $f^{-1}([a,b])$ is a union of intervals whose endpoints lie in the sets $f^{-1}(a)$ and $f^{-1}(b)$: thus, computing the inverse image reduces to solving the equations $f(y) = a$ and $f(y) = b$, arranging the solutions in increasing order, and then determining which of the resulting intervals are mapped into $[a,b]$ by f.
  - Here is a geometric picture of this procedure for computing the inverse image of $[1,2]$ under the function $f(x) = x^3 - 3x + 1$:



- Example: Find three intervals lying in the attracting basin of the attracting fixed point $x_0 = 0$ for the function $p(x) = \frac{1}{2}x + 3x^2 - 4x^3 + x^4$.
  - Clearly 0 is an attracting fixed point. As our starting point, we look for values of x for which $\left|\frac{p(x) - 0}{x}\right| < 1$: namely, with $\left|\frac{1}{2} + 3x - 4x^2 + x^3\right| < 1$.
  - Solving this inequality numerically gives three intervals, which are $(-0.336, 0.237)$, $(0.684, 1.675)$, and $(2.661, 3.078)$. We want the largest interval containing $x_0 = 0$ that is symmetric about 0 contained in one of those intervals, so we take $I = (-0.237, 0.237)$, rounded to three decimal places.
  - Now we numerically compute $p^{-1}(I)$, which is a union of three intervals, which we have rounded inward to three decimal places: $(-0.300, 0.237)$, $(1.138, 1.307)$, and $(2.894, 2.925)$. These three intervals all lie in the attracting basin.

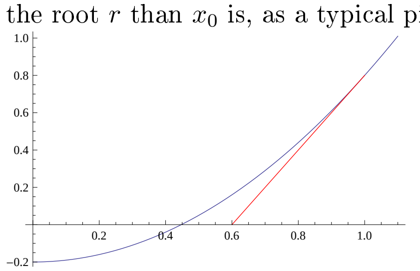
- We could continue this process and compute p^{-1} of each of these intervals: we end up with seven intervals $(-0.717, -0.715)$, $(-0.538, -0.513)$, $(-0.300, 0.237)$, $(1.138, 1.327)$, $(2.890, 2.925)$, $(2.980, 2.989)$, $(3.070, 3.071)$.
- If we continue computing the inverse images, the union of the resulting infinite number of intervals will be the full basin of attraction. Here is a plot of the results of five iterations of the inverse image map, starting with the initial interval I:



- Notice that the immediate basin appears to have endpoints roughly given by -0.300 and 0.237. Indeed, $p(-0.300) = 0.237$ and $p(0.237) = 0.237$, so one endpoint of the immediate basin is a fixed point and the other is one of its preimages (which is indeed one of the possibilities given by our theorem about the immediate basin).

1.4 Newton's Method

- Newton's method is an algorithm that (attempts to) give a numerical approximation of a zero of a differentiable function f.
  - It is immediately evident that any root-finding algorithm provides us with a way to compute the locations of fixed points and (pre)periodic points of functions numerically.
  - Newton's method also provides us with another collection of dynamical systems to study, and we can apply some of our techniques to analyze the results.
- The method is as follows: we begin at some starting point x_0. Then we draw the tangent line at x_0 to $y = f(x)$ and set x_1 to be the x-intercept of the tangent line. Now we iterate the process, by setting x_n to be the x-intercept of the tangent line at $x = x_{n-1}$ to $y = f(x)$, for each $n \geq 2$.
  - The idea is that, if x_0 is close to the root r, then the tangent line is a good approximation to the function $y = f(x)$, so the x-intercept of the tangent line (which is easy to compute) will, hopefully, be closer to the root r than x_0 is, as a typical picture suggests will be the case:



  - The tangent line has equation $y - f(x_0) = f'(x_0) \cdot (x - x_0)$, so the x-intercept is $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.
  - Thus, the points given by Newton's method are the same as the points in the orbit of x_0 under the map $N(x) = x - \dfrac{f(x)}{f'(x)}$.
- Definition: If $f(x)$ is a differentiable function, the Newton iteration function $N(x)$ is defined as $N(x) = x - \dfrac{f(x)}{f'(x)}$, and Newton's method is the result of computing the orbit of a point x_0 under $N(x)$.
  - Observe that, as long as f is always defined and $f'(x) \neq 0, \infty$, the fixed points of the Newton iteration function are the same as the zeroes of f.

- Example: Use Newton's method to approximate the value of $\sqrt{2}$.
  - By definition, $\sqrt{2}$ is the positive root of $f(x) = x^2 - 2$.
  - Thus, the Newton iteration function is $N(x) = x - \dfrac{f(x)}{f'(x)} = \dfrac{x}{2} + \dfrac{1}{x}$.
  - The orbit of 1 under N is 1, 1.5, 1.416667, 1.414216, 1.414214, 1.414214, ...
  - The orbit of 3 under N is 3, 1.833333, 1.462121, 1.414998, 1.414214, 1.414214, ...
  - We can see that the algorithm converges quite rapidly.
- Example: Use Newton's method to approximate the fixed point of $\cos(x)$.
  - We want to find a zero of the function $f(x) = \cos(x) - x$.
  - Thus, the Newton iteration function is $N(x) = x + \dfrac{\cos(x) - x}{\sin(x) + 1}$.
  - The orbit of 0 under N is 0, 1, 0.750364, 0.739113, 0.739085, 0.739085, ...
  - So we see the fixed point is approximately 0.739085.
- Example: Use Newton's method to approximate the real root of $f(x) = x^3 - 2x - 5$.
  - Notice that $f(2) = -1$ and $f(3) = 16$, so f has a root in $(2, 3)$ by the Intermediate Value Theorem. (Using some calculus, we can show that this function only has one real root.)
  - The Newton iteration function is $N(x) = x - \dfrac{f(x)}{f'(x)} = x - \dfrac{x^3 - 2x - 5}{3x^2 - 2} = \dfrac{2x^3 + 5}{3x^2 - 2}$.
  - The orbit of 2 under N is 2, 2.1, 2.094568, 2.094552, 2.094552, ... The root appears to have numerical value 2.094552.
  - The orbit of 3 under N is 3, 2.36, 2.127197, 2.095136, 2.094552, 2.094552, ... This orbit also approaches the root.
  - The orbit of 0 under N is 0, -2.5, -1.567, -0.503, -3.821, -2.549, -1.608, -0.576, ... The orbit does not appear to be converging to the real root, since we chose an initial point that was too far away.
- Example: Use Newton's method to approximate the real root of $f(x) = x^3 - 4x + 2$ lying in $(1, 2)$.
  - Notice that $f(1) = -1$ and $f(2) = 2$, so f does have a root in $(1, 2)$ by the Intermediate Value Theorem. In fact f has three real roots: one in $(-3, -2)$, one in $(0, 1)$, and one in $(1, 2)$.
  - The Newton iteration function is $N(x) = x - \dfrac{f(x)}{f'(x)} = x - \dfrac{x^3 - 4x + 2}{3x^2 - 4}$.
  - The orbit of 1 under N is 1, 0, 0.5, 0.538462, 0.539189, 0.539189, ... This does converge to a root of f, but not the one we were looking for!
  - The orbit of 2 under N is 2, 1.75, 1.680723, 1.675166, 1.675131, 1.675131, ... This does converge to the root we were looking for.
  - For completeness, of course, we could also use Newton's method to find the last root: the orbit of -2 is -2, -2.25, -2.215084, -2.214320, -2.214320, ...
- Example: Try to find the real root of $f(x) = x^{1/3}$ using Newton's method. (Of course, the root is clearly $x = 0$.)
  - The Newton iteration function is $N(x) = x - \dfrac{f(x)}{f'(x)} = -2x$.
  - The orbit of 0.1 under N is 0.1, -0.2, 0.4, -0.8, 1.6, -3.2, 6.4, ...
  - Notice that this orbit does not converge to 0. In fact, we can see immediately that 0 is a repelling fixed point for N, so no nearby orbit will converge to the real root of f.
  - Ultimately, the problem in this example is that $f'(0)$ is infinite.

- We would naturally like to know under what conditions a given fixed point of the Newton iteration function will be attracting.
  - From our previous results, if the fixed point is attracting, then the convergence of nearby orbits will be (at least) exponentially fast, with rate dictated by the value of N' at the fixed point.
  - If the value x_0 is a multiple root of f, then the analysis can be a bit trickier.
- Definition: If x_0 is a root of the continuous function f, the multiplicity of x_0 (as a root of f) is the smallest positive k such that there exists a continuous function $g(x)$ such that $f(x) = (x - x_0)^k \cdot g(x)$ and $g(x_0) \neq 0$, if such a k exists. (If there is no largest k such that $f(x) = (x - x_0)^k \cdot g(x)$ for a continuous function g, we say the multiplicity is ∞.)
  - The multiplicity of a root of a general function agrees with the usual sense of "multiple root" when referring to polynomials: for example, 1 is a root of multiplicity 2 for the function $(x^2 + 1)(x - 1)^2$ and of multiplicity 3 for the function $x(x - 1)^3$.
  - Example: If $f(x) = x^{4/3}$, then $x_0 = 0$ is a root of multiplicity $4/3$.
  - Example: If $f(x) = 0$ is the identically zero function, then $x_0 = 0$ is a root of infinite multiplicity.
  - Most reasonable functions will only have roots of finite multiplicity. The standard example of a nontrivial function having a root of infinite multiplicity is $f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$: all the derivatives of f are zero at $x = 0$.
- Proposition: If x_0 is a root of f of multiplicity k, then $f^{(d)}(x_0) = 0$ for all $d < k$. Furthermore, if $k \geq 1$ is an integer, then x_0 is a root of f of multiplicity k if and only if $f^{(d)}(x_0) = 0$ for all $d < k$ and $f^{(k)}(x_0)$ is nonzero and finite.
  - This proposition provides an easy way to compute the multiplicity of a root for a differentiable function: for example, if $f(x) = \sin(x)$, then $x_0 = k\pi$ is a root of multiplicity 1 for each integer k, since the derivative $f'(x_0)$ is nonzero at each such point.
  - Remark: Note the similarity to the statement of the classification of neutral fixed points. (Indeed, the k from that theorem is the multiplicity of the value x_0 as a root of the function $f(x) - x$.)
  - Proof: If $f(x) = (x - x_0)^k g(x)$, then applying the product rule shows that $f^{(d)}$ is a sum of terms involving the first d derivatives of $(x - x_0)^k$ and $g(x)$. For $d < k$ all of the derivatives of $(x - x_0)^k$ are zero, so we see $f^{(d)}(x_0)$ for $d < k$, giving the first statement. Also, if $d = k$ then we will get a single term $k! \cdot g(x_0)$, so $f^{(k)}(x_0) = k! \cdot g(x_0)$ is nonzero since $g(x_0) \neq 0$.
  - Conversely, if x_0 is a root of f of integral multiplicity k, then by k applications of L'Hôpital's rule we see that $\lim_{x \to x_0} \dfrac{f(x)}{(x - x_0)^k} = f^{(k)}(x_0)$, so the function $\dfrac{f(x)}{(x - x_0)^k}$ (defined for $x \neq x_0$) can be extended to be continuous and nonzero at $x = x_0$. We can then simply take $g(x)$ to be the resulting continuous function.
- The multiplicity of a root will control how fast Newton's method will converge near that root:
- Theorem (Newton's Fixed Point Theorem): Suppose f is continuously differentiable and N is its Newton iteration function. If x_0 is a root of f of finite multiplicity $k \geq 1$, then x_0 is an attracting fixed point of N, and if x_0 is a root of multiplicity $k = 1$, then x_0 is a superattracting fixed point of N.
  - Proof: By definition, x_0 will be an attracting fixed point of N if $|N'(x_0)| < 1$, and it will be superattracting if $N'(x_0) = 0$. We also note that because f' is continuous, there are no points where f' is ∞, so the only fixed points of N are the zeroes of f.
  - By the quotient rule we see that $N'(x) = \dfrac{f(x)f''(x)}{[f'(x)]^2}$ whenever $f'(x) \neq 0$.
  - So if $f'(x_0) \neq 0$, which occurs if x_0 has multiplicity 1, we immediately see that $N(x_0) = x_0$ and $N'(x) = 0$, so that x_0 is a superattracting fixed point of N.
  - If $f'(x_0) = 0$ and x_0 has multiplicity $k > 1$, then by the proposition above we can write $f(x) = (x - x_0)^k g(x)$ for a function g with $g(x_0) \neq 0$. To ease notation, make the change of variables $y = x - x_0$ to move the fixed point to zero: then $f(y) = y^k g(y)$ where $g(0) \neq 0$.

- Then $f'(y) = ky^{k-1}g(y) + y^k g'(y)$ and $f''(y) = k(k-1)y^{k-2}g(y) + 2ky^{k-1}g'(y) + y^k g''(y)$, so after some algebra we see that $N(y) = y - \dfrac{y\,g(y)}{k\,g(y) + g'(y)}$, so that $N(0) = 0$.
- Furthermore, $N'(y) = \dfrac{k(k-1)g(y)^2 + 2kyg'(y)g(y) + y^2 g''(y)g(y)}{k^2 g(y)^2 + 2kyg(y)g'(y) + y^2 g'(y)^2}$, so $N'(0) = \dfrac{k(k-1)g(y)^2}{k^2 g(y)^2} = 1 - \dfrac{1}{k}$, since $g(y) \neq 0$ by assumption.
- Since this quantity has absolute value less than 1 as long as $k \geq 1$, we see that $y = 0$ (i.e., $x = x_0$) is an attracting fixed point of N as claimed.

• Newton's fixed point theorem guarantees that (as long as f does not have any zeroes of infinite or undefined multiplicity) each of the zeroes of f will show up as an attracting fixed point of N, and that these are the only fixed points of N.

- A natural question to ask is: what does the attracting basin for each fixed point of N look like?
- A fuller discussion of this topic belongs properly to a numerical analysis course, but from our results about attracting points, we can say a few things.
- For example, the immediate basin for each fixed point will contain the interval on which $|N'(x)| = \left|\dfrac{f(x)f''(x)}{f'(x)^2}\right| < 1$. (Though this function is rather hard to analyze, as we just saw.)
- Also, the endpoint of any fixed point's immediate basin cannot be another fixed point, because every fixed point is attracting. Thus, each fixed point's immediate basin either has endpoints that form a 2-cycle under N, or has endpoints that are $\pm\infty$ or points where N is undefined (i.e., zeroes of f').
- We also remark that by the mean value theorem, f' will have a zero between any two zeroes of f, N will always be undefined somewhere in the interval between any two attracting fixed points.
- In general, the full attracting basin can be quite complicated (as with attracting basins of general functions).

• If f does not have any roots at all, the Newton iteration function N has no fixed points: but this does not mean its dynamics are uninteresting.

• Example: Try to find a real root of $f(x) = x^2 + 1$ using Newton's method. (Of course, f has no real roots.)

- The Newton iteration function is $N(x) = x - \dfrac{f(x)}{f'(x)} = \dfrac{x}{2} - \dfrac{1}{2x}$.
- The orbit of 0.1 under N is 0.1, -4.95, -2.37399, -0.97638, 0.02391, -20.90272, -10.42744, -5.16577, -2.48609, -1.04193, -0.04198, 12.14959, ...
- The orbit of 0.2 under N is 0.2, -2.4, -0.99167, 0.00837, -59.7477, -29.86402, -14.91527, -7.42411, -3.64471, -1.68517, -0.54588, 0.64302, -0.45608, ...
- These orbits, of course, will not approach a fixed point, since N has no fixed points.
- It is not hard to show that orbits will behave as follows: orbits far from 0 will approach zero monotonically until they land in the interval $(-1, 1)$, at which point they will switch sign after each iteration until they land inside $(1 - \sqrt{2}, \sqrt{2} - 1)$, where the next iteration will carry them outside $(-1, 1)$ and the process will repeat.

Well, you're at the end of my handout. Hope it was helpful.