Complex Analysis (part 3): Complex Integration (by Evan Dummit, 2022, v. 1.00)

Contents

3 Complex Integration

In this chapter, our goal is to study integration in the complex plane. We extend the notion of a line integral of a real-valued function along a curve in $\mathbb{R}^2$ to define the line integral of a complex function along a curve in $\mathbb{C}$, and then characterize various properties of line and contour integrals in $\mathbb{C}$. We then develop Cauchy's integral theorem and Cauchy's integral formula, which together form one of the most fundamental tools in complex analysis, and study many of their immediate consequences.

3.1 Integrals on Complex Curves

- We would like to develop a complex analogue to integration, to parallel our complex derivative.
  - One possibility is simply to generalize the functions we allow ourselves to integrate on an interval $[a, b]$: in other words, to define the integral $\int_a^b f(t)\,dt$ where now $f(t)$ is a complex-valued function.
  - We can do this just by writing the usual definition of a real integral using Riemann sums but now allowing the function $f(t) = x(t) + iy(t)$ to be complex.
  - However, it is not hard to see that since Riemann sums and real integrals are linear, this simply amounts to defining $\int_a^b [x(t) + iy(t)]\,dt = \int_a^b x(t)\,dt + i \int_a^b y(t)\,dt$.
  - In particular, by an immediate application of the fundamental theorem of calculus for real-valued functions to the real and imaginary parts, we see that if $F(t)$ is differentiable with $F'(t) = f(t)$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.
- A far more useful generalization of integration is to integrate a complex function f along a complex curve γ.

3.1.1 Complex Line Integrals via Riemann Sums

- Just as with the integral of a real-valued function, we can formally define complex line integrals as a limit of Riemann sums (though, just as with integrals of real-valued functions, we will essentially never actually use Riemann sums to evaluate integrals).
  - Explicitly, suppose that $\gamma : [a, b] \to \mathbb{C}$ is a continuous curve, so that $\lim_{t\to t_0} \gamma(t) = \gamma(t_0)$ for all $t \in [a, b]$, with the appropriate one-sided limits taken at the endpoints, and let $f : \mathbb{C} \to \mathbb{C}$ be a complex function.
  - For convenience, we will set up Riemann sums using a partition of the interval $[a, b]$ rather than a partition of the actual geometric curve $\mathrm{im}(\gamma)$.
- <u>Definition</u>: A <u>tagged partition</u> of the interval $[a, b]$ is a set $P^* = \{t_0, t_1, \dots, t_n\}$ with $a = t_0 < t_1 < \cdots < t_n = b$, together with values $t_i^* \in [t_{i-1}, t_i]$ for each $1 \le i \le n$. By applying γ to all of the t_i and t_i^*, we obtain a tagged partition of the curve γ.
- <u>Definition</u>: If P^* is a tagged partition of $[a, b]$, the associated <u>Riemann sum</u> is $\mathrm{RS}_{P^*}(f) = \sum_{k=1}^{n} f(z_i^*)\Delta z_i$ where $z_i^* = \gamma(t_i^*)$ represents the point on γ at which we evaluate f, and $\Delta z_i = \gamma(t_i) - \gamma(t_{i-1})$ represents the segment on which we are summing. The <u>norm</u> of the partition is defined to be the maximum of the values $|\Delta z_i|$ for $1 \le i \le n$.
- <u>Definition</u>: We say f <u>has integral L on γ</u>, and write $\int_\gamma f(z)\,dz = L$, if for every $\epsilon > 0$ there exists a $\delta > 0$ (depending on ϵ) such that for every tagged partition P^* with $\mathrm{norm}(P^*) < \delta$, we have $|RS_P(f) - L| < \epsilon$.
  - Intuitively, the idea is that we define the integral as the limit of the value of the Riemann sums as the norm tends to 0 (i.e., as the pieces of the partition of γ become uniformly small).
  - By some fairly tedious analysis of refinements of partitions[1], one may prove that if f is continuous on γ (meaning that for any $z_0 \in \gamma$ and any $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for all $z \in \gamma$ such that $|z - z_0| < \delta$) and γ is continuously differentiable, then in fact f is integrable on γ.
- For no additional cost we can weaken slightly the differentiability requirement of the curves over which we set up complex line integrals:
- <u>Definition</u>: A <u>continuous curve</u> γ in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$. Explicitly, this means $\gamma(t) = x(t) + iy(t)$ for some continuous real-valued functions $x(t)$ and $y(t)$ on the interval $[a, b]$.
- <u>Definition</u>: A continuous curve is <u>closed</u> if $\gamma(a) = \gamma(b)$, and it is <u>simple</u> if γ is one-to-one except for having $\gamma(a) = \gamma(b)$.
- <u>Definition</u>: A continuous curve is <u>rectifiable</u> if γ is continuously differentiable except at a finite number of points; as part of this requirement we also require the one-sided limits $\lim_{t\to a+} \gamma'(t)$ and $\lim_{t\to b-} \gamma'(t)$ to exist.
  - The motivation for the endpoint conditions in the definition of rectifiability is to ensure that the fundamental theorem of calculus holds on $[a, b]$ for the function γ: explicitly, we have $\gamma(b) - \gamma(a) = \lim_{\epsilon\to 0+}[\gamma(b - \epsilon) - \gamma(a + \epsilon)] = \lim_{\epsilon\to 0+} \int_{a+\epsilon}^{b-\epsilon} \gamma'(t)\,dt = \int_a^b \gamma'(t)\,dt$.
  - If f is continuous and γ is rectifiable, then by an appropriate partition refinement argument, we see that f is integrable on γ and the integral of f is the sum of $\int f(z)\,dz$ over the finitely many pieces where γ is continuously differentiable.
  - It is also easy to see that if $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [b, c] \to \mathbb{C}$ are rectifiable with $\gamma_1(b) = \gamma_2(b)$, then the union $\gamma_1 \cup \gamma_2 : [a, c] \to \mathbb{C}$ defined by $\gamma_1 \cup \gamma_2(t) = \begin{cases} \gamma_1(t) & \text{for } a \le t \le b \\ \gamma_2(t) & \text{for } b < c \le c \end{cases}$ is rectifiable. Geometrically, this corresponds to "gluing" γ_2 to the end of γ_1.

[1] The usual approach for real-valued functions is to consider for a fixed P^* the "upper sum" (using the selection of tag points maximizing the Riemann sum for a fixed partition P^*) and the "lower sum" (minimizing the Riemann sum), then to show that the upper sum decreases and the lower sum increases when refining a partition, and finally using (uniform) continuity of f on γ to show that for an appropriate refinement the difference between the upper and lower sums can be made arbitrarily small. Applying the result to the real and imaginary parts of a complex integral, and making an estimate on the error obtained by the fact that partition refinement does not completely respect taking real and imaginary parts (this is where the continuous differentiability of γ is needed) yields the result for complex integrals.

- In most of our actual examples we will integrate over smooth curves, but it is convenient to be able to use rectifiable curves, since for example the boundary of a triangle or rectangle is non-differentiable at each of the vertices, but is still rectifiable.

- Using the definition in terms of Riemann sums, we obtain various basic properties of complex integrals analogous to those for real integrals.

- Proposition (Basic Integral Properties): Suppose that $\gamma : [a, b] \to \mathbb{C}$ is rectifiable and f and g are integrable on γ. Then the following hold:

  1. For any $c \in \mathbb{C}$ the scalar multiple cf is integrable on γ and $\int_\gamma (cf)(z)\,dz = c \int_\gamma f(z)\,dz$.
  2. The sum $f+g$ and difference $f-g$ are both is integrable on γ and $\int_\gamma (f \pm g)(z)\,dz = \int_\gamma f(z)\,dz \pm \int_\gamma g(z)\,dz$.
  3. If γ_1 and γ_2 are rectifiable and f is integrable on both, then f is integrable on $\gamma_1 \cup \gamma_2$ and $\int_{\gamma_1 \cup \gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$.
  4. If $-\gamma$ is the reverse of γ, defined by $(-\gamma)(t) = \gamma(a+b-t)$, then $\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$.
  5. If f is continuous then $\left| \int_a^b f(t)\,dt \right| \le \int_a^b |f(t)|\,dt$.

     - Note that (3) is the analogue of $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ while (4) is the analogue of $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.
     - The triangle inequality (5) does not hold on arbitrary curves γ, since the second integral $\int_\gamma |f(z)|\,dz$ need not even be real, which is why we only pose it for integrals along the real line.
     - Proofs: (1) follows by noting that Riemann sums for cf are simply c times a Riemann sum for f.
     - (2) follows by noting that a Riemann sum for $f + g$ is the sum of a Riemann sum for f with a Riemann sum for g.
     - (3) follows by noting that a Riemann sum for f on $\gamma_1 \cup \gamma_2$ is a Riemann sum for f on γ_1 plus a Riemann sum on γ_2, provided the transition point $t = b$ is included in the partition. Since we may freely refine partitions, this assumption causes no issues.
     - (4) follows by noting that a Riemann sum for $-\gamma$ is a Riemann sum for γ with all terms scaled by -1 (using the same tag points but interchanging t_i and t_{i-1}).
     - (5) follows by applying the usual triangle inequality in $\mathbb{C}$ to a Riemann sum for $\int_a^b f(t)\,dt$ and observing that the result is a Riemann sum for $\int_a^b |f(t)|\,dt$.

3.1.2 Evaluating Complex Line Integrals

- We would like to actually evaluate some complex integrals, which is quite unpleasant using the Riemann sum definition. Conveniently, we can avoid most of the technical annoyances by converting a complex integral to an integral over a real variable, in the same way that we can convert a line integral on a real plane curve into a single integral of a real variable.

  - Explicitly, suppose P is a given partition of $[a, b]$. Under the assumption that $\gamma(t)$ is differentiable, we have $\Delta z_i = \gamma(t_i) - \gamma(t_{i-1}) \approx (t_i - t_{i-1})\gamma'(t_i)$.
  - Therefore, the Riemann sum $\sum_{k=1}^n f(z_{i-1})\Delta z_i$ for $f(z)$ is approximately equal to $\sum_{k=1}^n f(\gamma(t_{i-1}))\,\gamma'(t_{i-1})\,(t_i - t_{i-1})$, which is in turn a Riemann sum for the integral $\int_a^b f(\gamma(t))\,\gamma'(t)dt$ with tag points $t_i^* = t_{i-1}$.
  - This latter integral is much easier to evaluate, since we may just integrate the real and imaginary parts on $[a, b]$ separately, as we discussed earlier.
  - As we will show, under the assumption that $\gamma(t)$ is continuously differentiable, the difference between these two Riemann sums must go to zero as the norm of the partition goes to zero.

- Proposition (Complex Line Integrals): Suppose $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable complex curve and $f(z)$ is a continuous function on γ. Then f is integrable on γ and $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$.

- ◦ Proof: Since f is integrable as noted earlier and $f(\gamma(t))\,\gamma'(t)$ is integrable on $[a,b]$ since it is continuous, we just need to show that the Riemann sums for f approach Riemann sums for $\int_a^b f(\gamma(t))\,\gamma'(t)\,dt$. It suffices to show that this holds on each of the finitely many subintervals on which γ is continuously differentiable.
  - ◦ More precisely, for any $\epsilon > 0$ we will show that the left-endpoint Riemann sums for the first integral and the second integral with sufficiently small norm are within ϵ of one another.
  - ◦ Since f is continuous on the closed curve γ, the function $g(t) = |f(\gamma(t))|$ from $[a,b] \to \mathbb{R}$ is continuous on the closed interval $[a,b]$, so by the extreme value theorem it is bounded above: then $|f| \le M$ on γ for some M.
  - ◦ Also since γ' is continuous on $[a,b]$ it is uniformly continuous by the Heine-Cantor theorem applied to both the real and imaginary parts, so there exists $\delta > 0$ such that $|s-t| < \delta$ implies $|\gamma'(s) - \gamma'(t)| < \epsilon/[(b-a)M]$.
  - ◦ Now let $P = \{t_0, \dots, t_n\}$ be any partition of $[a,b]$ of norm less than δ and take $t_i^* = t_{i-1}$.
  - ◦ By uniform continuity, the absolute value of the difference between $\gamma(t_i) - \gamma(t_{i-1})$ and $(t_i - t_{i-1})\gamma'(t_i)$ is bounded above by $(t_{i-1}-t_i)\epsilon/[(b-a)M]$, so the difference between the Riemann sum $\sum_{k=1}^n f(\gamma(t_{i-1}))(\gamma(t_i) - \gamma(t_{i-1}))$ and the Riemann sum $\sum_{k=1}^n f(\gamma(t_{i-1}))\,\gamma'(t_{i-1})\,(t_i - t_{i-1})$ is bounded in absolute value by $\sum_{k=1}^n M(t_{i-1} - t_i)\epsilon/[(b-a)M] = \epsilon$.
  - ◦ Hence for any partition of norm at most δ we see that the Riemann sums are within ϵ, so taking the limit shows that the difference between the integrals is at most ϵ. This holds for all $\epsilon > 0$, so the integrals are equal.

- Example: For $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$, evaluate the complex line integral $\int_\gamma z^{-1}\,dz$.
  - ◦ We simply compute $\gamma'(t) = ie^{it}$ and $f(z) = z^{-1} = e^{-it}$.
  - ◦ Thus $\int_\gamma z^{-1}\,dz = \int_0^{2\pi} e^{-it} \cdot ie^{it}\,dt = \int_0^{2\pi} i\,dt = \boxed{2\pi i}$.

- Example: For $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$, evaluate the complex line integral $\int_\gamma z^n\,dz$ for each integer $n \ne -1$.
  - ◦ As above we have $\gamma'(t) = ie^{it}$ and $f(z) = z^n = (e^{it})^n$.
  - ◦ Thus $\int_\gamma z^n\,dz = \int_0^{2\pi} (e^{it})^n \cdot ie^{it}\,dt = \int_0^{2\pi} ie^{(n+1)it}\,dt = \left.\dfrac{e^{(n+1)it}}{n+1}\right|_{t=0}^{2\pi} = \dfrac{1}{n+1} - \dfrac{1}{n+1} = \boxed{0}$.

- Example: For $\gamma(t) = t^2 + it$ for $0 \le t \le 1$, evaluate the complex line integral $\int_\gamma (1-2z)\,dz$.
  - ◦ We have $\gamma'(t) = 2t + i$ and $f(z) = 1 - 2(t^2 + it) = (1 - 2t^2) - 2it$.
  - ◦ Thus $\int_\gamma (1-2z)\,dz = \int_0^1 [(1-2t^2) - 2it](2t+i)\,dt = \int_0^1 [(4t - 4t^3) + (1 - 6t^2)i]\,dt = (2t - t^4) + (t - 2t^3)i\big|_{t=0}^1 = \boxed{1-i}$.

- Example: For $\gamma(t) = (1+t) + (2-t)i$ for $0 \le t \le 3$, evaluate the complex line integral $\int_\gamma z^2\,dz$.
  - ◦ We have $\gamma'(t) = 1 - i$ and $f(z) = [(1+t) + (2-t)i]^2$.
  - ◦ Thus $\int_\gamma z^2\,dz = \int_0^3 [(1+t) + (2-t)i]^2(2t+i)\,dt = \int_0^3 [(1 + 8t - 2t^2) + (7 - 4t - 2t^2)i]\,dt = (t + 4t^2 - \frac{2}{3}t^3) + (7t - 2t^2 - \frac{2}{3}t^3)i\big|_{t=0}^3 = \boxed{21 - 15i}$.

- Example: For $\gamma(t) = t + 2it$ for $0 \le t \le 1$, evaluate the line integrals $\int_\gamma \mathrm{Re}(z^2)\,dz$ and $\int_\gamma \mathrm{Im}(z^2)\,dz$.
  - ◦ We have $\gamma'(t) = 1 + 2i$ and $\mathrm{Re}(z^2) = \mathrm{Re}(-3t^2 + 4it^2) = -3t^2$.
  - ◦ Thus $\int_\gamma \mathrm{Re}(z^2)\,dz = \int_0^1 (-3t^2)(1+2i)\,dt = -(1+2i)t^3\big|_{t=0}^1 = \boxed{-1-2i}$.
  - ◦ Also as $\mathrm{Im}(z^2) = 4t^2$, we see $\int_\gamma \mathrm{Im}(z^2)\,dz = \int_0^1 (4t^2)(1+2i)\,dt = \frac{4}{3}(1+2i)t^2\big|_{t=0}^1 = \boxed{\frac{4}{3} + \frac{8}{3}i}$.

- Although we refer to the function γ as a “curve”, in fact the value of the integral does not depend on the particular parametrization we use for the curve.

- For example, consider the line integral of $f(z) = z^2$ on the upper half of the unit circle traversed counterclockwise from $z = 1$ to $z = -1$.
- If we use the parametrization $\gamma_1(t) = e^{it}$ for $0 \le t \le \pi$, the integral is $\int_{\gamma_1} z^2\, dz = \int_0^\pi (e^{it})^2 \cdot ie^{it}\, dt = \int_0^\pi ie^{3it}\, dt = \left.\dfrac{e^{3it}}{3}\right|_{t=0}^{\pi} = -\dfrac{2}{3}$.
- If instead we use the parametrization $\gamma_2(s) = e^{i\sqrt{s}}$ for $0 \le s \le \pi^2$, which follows the same path but at non-constant speed, the integral is $\int_{\gamma_2} z^2\, dz = \int_0^{\pi^2} (e^{i\sqrt{s}})^2 \cdot i\dfrac{1}{2\sqrt{s}} e^{i\sqrt{s}}\, dt = \int_0^{\pi^2} \dfrac{i}{2\sqrt{s}} e^{3i\sqrt{s}}\, dt = \left.\dfrac{e^{3i\sqrt{s}}}{3}\right|_{t=0}^{\pi^2} = -\dfrac{2}{3}$.
- Indeed, once we actually plug in the parametrizations, the two resulting integrals are easily seem to be equal by making the change of variables $t = \sqrt{s}$, which (unsurprisingly) also shows that the parametrizations themselves describe the same curve. This result holds in general:

- Proposition (Reparametrization): Suppose that $f(z)$ is continuous on the rectifiable curve $\gamma_1 : [a, b] \to \mathbb{C}$, and $\gamma_2 : [c, d] \to \mathbb{C}$ is also rectifiable and there exists a continuous one-to-one function $g : [a, b] \to [c, d]$ such that $\gamma_1 = \gamma_2 \circ g$. Then $\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$.
  - Proof: Since both integrals exist it suffices to observe that Riemann sums on γ_2 yield Riemann sums on γ_1. But this follows immediately: if P^* is a tagged partition of $[a, b]$, then applying g yields a tagged partition Q^* of $[c, d]$, and the underlying Riemann sums are then equal since $\gamma_1 = \gamma_2 \circ g$.

- As a consequence of our reparametrization result, we only need a description of the actual curve γ in $\mathbb{C}$ to evaluate a complex line integral. We may therefore refer to such a contour integral merely by describing the curve ("contour") and function to be integrated on it.
  - We can parametrize the line segment from z_0 to z_1 via $\gamma(t) = (1-t)z_0 + tz_1$ for $0 \le t \le 1$; note $\gamma(0) = z_0$ and $\gamma(1) = z_1$.
  - We can parametrize the circle of radius r centered at z_0 oriented counterclockwise via $\gamma(t) = z_0 + re^{it}$ for $0 \le t \le 2\pi$; note that $\gamma(0) = \gamma(2\pi) = z_0 + r$. Additionally, if we want only a portion of the circle, or to have γ go around the circle multiple times, we need only adjust the range of values for t.

- Example: Find $\int_\gamma \overline{z}^2\, dz$ where γ is the counterclockwise boundary of the triangle with vertices -1, 1, and i.
  - There are three boundary components to parametrize.
  - For the segment from -1 to 1 we take $\gamma(t) = -1 + 2t$ for $0 \le t \le 1$. Then $\gamma'(t) = 2$ and $\overline{z}^2 = (-1 + 2t)^2$ so the integral is $\int_0^1 (-1 + 2t)^2 \cdot 2\, dt = \int_0^1 (2 - 8t + 8t^2)\, dt = 2/3$.
  - For the segment from 1 to i we take $\gamma(t) = (1 - t) + it$ for $0 \le t \le 1$. Then $\gamma'(t) = -1 + i$ and $\overline{z}^2 = [(1-t) - it]^2$ so the integral is $\int_0^1 [(1-t) - it]^2 \cdot (-1+i)\, dt = \int_0^1 [(-1+4t-2t^2) + (1-2t^2)i] = 1/3 + i/3$.
  - For the segment from i to -1 we take $\gamma(t) = -t + (1 - t)i$ for $0 \le t \le 1$. Then $\gamma'(t) = -1 - i$ and $\overline{z}^2 = [-t + (1-t)i]^2$ so the integral is $\int_0^1 [-t + (1-t)i]^2 \cdot (-1-i)\, dt = \int_0^1 [(1-2t^2) + (1-4t-2t^2)i] = 1/3 - i/3$.
  - Therefore the integral around the entire boundary is the sum, which is $\boxed{4/3}$.

- Example: Find $\int_\gamma \mathrm{Im}(z)\, dz$ where γ is the curve that goes around the circle $|z - i| = 2$ three times counterclockwise.
  - The parametrization $\gamma(t) = i + 2e^{it}$ for $0 \le t \le 6\pi$ winds around the given circle three times counterclockwise.
  - Then $\gamma'(t) = 2ie^{it}$ and $\mathrm{Im}(z) = \mathrm{Im}(i + 2e^{it}) = 1 + 2\sin t = 1 + ie^{-it} - ie^{it}$, so the desired integral is $\int_0^{6\pi} [1 + ie^{-it} - ie^{it}](2ie^{it})\, dt = \int_0^{6\pi} [2ie^{it} - 2 + 2e^{2it}]\, dt = [2e^{it} - 2t - ie^{2it}]\Big|_{t=0}^{6\pi} = \boxed{-12\pi}$.

3.1.3 Antiderivatives, The Fundamental Theorem of Calculus

- It is natural to expect that there should be complex analogues of our familiar properties of derivatives, antiderivatives, and integrals (e.g., the fundamental theorem of calculus). We collect many of these properties here:

- Theorem (Properties of Antiderivatives and Line Integrals): Suppose that $f(z)$ is a continuous function on a region R and γ is a contour in R. Then the following hold:

  1. If R is connected and $f'(z) = 0$ on R, then f is constant on R.
     - Proof: First suppose $z, z' \in R$ and suppose that there is a continuously differentiable contour $\gamma : [a, b] \to \mathbb{C}$ that connects them in R.
     - Then the function $g : [a, b] \to \mathbb{C}$ given by $g(t) = f(\gamma(t))$ is differentiable so by the chain rule $g'(t) = f'(\gamma(t))\gamma'(t) = 0$ identically. But this means the real part and imaginary part of g are both constant (since they are differentiable real-valued functions with zero derivative), so g is constant. In particular $f(z) = g(a) = g(b) = f(z')$.
     - Now for general $z, z' \in R$ since R is connected there exists a path $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ where γ_i is continuously differentiable and starts at z_{i-1} and ends at z_i, from z_1 to z_2. Applying the result above to each γ_i shows $f(z) = f(z_1) = f(z_2) = \cdots = f(z_n) = f(z')$. Thus the value of f is the same at each point in R, meaning that f is constant.
  2. (Uniqueness of Antiderivatives) If F and G are holomorphic on R and $F'(z) = G'(z)$ on R, then $F(z) = G(z) + c$ for some $c \in \mathbb{C}$.
     - Proof: Apply (1) to $F(z) - G(z)$, which has derivative zero hence is constant on R.
  3. (Independence of Path) Suppose that $F(z)$ is holomorphic on a region R with $F'(z) = f(z)$ and that γ is any contour in R from z_0 to z_1. Then $\int_\gamma f(z)\,dz = F(z_1) - F(z_0)$.
     - Note that this is the analogue of the second part of the usual real fundamental theorem of calculus: if $F(x)$ is differentiable on $[a, b]$ with $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.
     - We refer to this result as "independence of path" (particularly in the context of real line integrals in the plane), since it states that the integral of $F'(z)$ from z_0 to z_1 depends only on the values of $F(z)$ at the endpoints, and not on the specific path of integration between them.
     - Proof: First suppose that γ is continuously differentiable, so that $\gamma : [a, b] \to R$ is continuously differentiable.
     - Then by the chain rule we have $\frac{d}{dt}[F(\gamma(t))] = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t)$.
     - Therefore $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b \frac{d}{dt}[F(\gamma(t))]\,dt = F(\gamma(b)) - F(\gamma(a))$ by the usual fundamental theorem of calculus.
     - For the general result for arbitrary contours, say $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ where γ_i is continuously differentiable and starts at z_{i-1} and ends at z_i, by our result on gluing curves we have $\int_\gamma f(z)\,dz = \sum_{i=1}^n \int_{\gamma_i} f(z)\,dz = \sum_{i=1}^n [F(z_i) - F(z_{i-1})] = F(z_n) - F(z_0)$ as required.
  4. (Necessity for Antiderivatives) Suppose that $F(z)$ is holomorphic on a region R with $F'(z) = f(z)$ and that γ is any closed contour in R. Then $\int_\gamma f(z)\,dz = 0$.
     - By changing focus from $F(z)$ to $f(z)$ here, we obtain a necessary condition for the existence of an antiderivative $F(z)$ of $f(z)$ on R: namely, for any closed contour γ in R, we must have $\int_\gamma f(z)\,dz = 0$.
     - Proof: By (3) we have $\int_\gamma f(z)\,dz = F(z_1) - F(z_0)$ where γ starts at z_0 and ends at z_1. But $z_0 = z_1$ for a closed contour, so $F(z_1) - F(z_0) = 0$.
  5. (Existence of Antiderivatives, I) Let R be a connected open set and $f(z)$ be a continuous function on R such that $\int_\gamma f(z)\,dz = 0$ for any closed contour in R. Choose any $a \in R$ and then for each $z \in R$ select any path γ in R from a to z. Then $F(z) = \int_\gamma f(z)\,dz$ is well defined and has $F'(z) = f(z)$ for all z.
     - Note that this is the analogue of the first part of the fundamental theorem of calculus for real functions, which states that if f is continuous and $a \in \mathbb{R}$, then $F(x) = \int_a^x f(t)\,dt$ is differentiable and has derivative $F'(x) = f(x)$.

- However, here we have the extra hypothesis that $\int_\gamma f(z)\,dz = 0$ for any path γ, which is necessary by (4). So the point of this result is that the condition of (4) is both necessary and sufficient for an antiderivative to exist.
- Proof: First, to show that $F(z_0)$ is well defined for each $z_0 \in R$, since R is connected (hence path-connected since it is open) there is at least one contour from a to z_0. Now suppose we have two paths γ_1 and γ_2 from a to z_0 in R.
- Then by setting $\gamma = \gamma_1 \cup (-\gamma_2)$, we see that $\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{-\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz$, and so $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$. Therefore, the value of $F(z_0)$ in our definition above does not depend on which path we take from a to z_0: this means $F(z_0)$ is well defined.
- Now for the derivative, since R is open and $z_0 \in R$, there exists some $r > 0$ such that the open disc $|z - z_0| < r$ is contained in R. We have $\dfrac{F(z_0+h) - F(z_0)}{h} = \dfrac{1}{h}\int_\gamma f(z)\,dz$, where the integral is taken along any path γ from z_0 to $z_0 + h$.
- Since we are only interested in the limit as $h \to 0$, and R is open, we may restrict attention to $|h| < r$. Then the line segment from z_0 to $z_0 + h$ is fully contained in the open disc $|z - z_0| < r$, so we may take γ to be that line segment, which is parametrized by $\gamma(t) = z_0 + ht$ for $0 \le t \le 1$.
- Then we have $\dfrac{F(z_0+h) - F(z_0)}{h} = \dfrac{1}{h}\int_\gamma f(z)\,dz = \dfrac{1}{h}\int_0^1 f(z_0+ht)\,h\,dt = \int_0^1 f(z_0+ht)\,dt$.
- Since f is continuous on the open disc, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z_0) - f(z)| < \epsilon$ for $|z - z_0| < \delta$. Then by the triangle inequality for integrals we have $\left|\int_0^1 [f(z_0+ht) - f(z_0)]\,dt\right| \le \int_0^1 |f(z_0+ht) - f(z_0)|\,dt < \epsilon$ under the hypothesis that $|h| < \delta$.
- Therefore, taking $h \to 0$ in $\int_0^1 f(z_0+ht)\,dt$ yields a value within ϵ of $f(z_0)$ for every $\epsilon > 0$, whence $\lim_{h\to 0} \int_0^1 f(z_0+ht)\,dt = f(z_0)$. Thus, we have $F'(z_0) = \lim_{h\to 0} \dfrac{F(z_0+h) - F(z_0)}{h} = \lim_{h\to 0} \int_0^1 f(z_0+ht)\,dt = f(z_0)$ as required.

- Example: Evaluate the complex line integral $\int_\gamma z^n\,dz$ for each integer $n \ne -1$, where γ is the line segment from 0 to i.
  - Note that for $f(z) = z^n$ we have an antiderivative $F(z) = \dfrac{z^{n+1}}{n+1}$.
  - Therefore, by applying independence of path, we see that $\int_\gamma z^n\,dz = \dfrac{z^{n+1}}{n+1}\Big|_{z=0}^{i} = \boxed{\dfrac{i^{n+1}}{n+1}}$.
  - Note that we could also compute the integral using a parametrization: with $\gamma(t) = it$ for $0 \le i \le 1$, we have $\gamma'(t) = i$ and $f(z) = z^n = (it)^n$, so the integral is $\int_0^1 (it)^n \cdot i\,dt = \dfrac{i^{n+1}t^{n+1}}{n+1}\Big|_{t=0}^{1} = \dfrac{i^{n+1}}{n+1}$, which is of course the same. The appeal of using path-independence is that we see that the value of the integral is independent of the shape of the actual contour γ.

- Interestingly, the existence (and nonexistence) of antiderivatives depends both on the function $f(z)$ and on the region R on which we are trying to construct an antiderivative.

- Example: Show that $f(z) = z^{-1}$ has no antiderivative on the annulus $1/2 < |z| < 2$.
  - As we calculated previously, the integral of $f(z)$ around the contour γ that winds once counterclockwise around the unit circle is $2\pi i \ne 0$.
  - Therefore, $f(z)$ cannot have an antiderivative on R, since if it did, the integral around every closed contour would be zero.
  - Remark: More generally we can show that $f(z)$ has no antiderivative on any region of the form $a < |z| < b$ by integrating around an appropriate circle $|z| = r$ for some $a < r < b$.

- Example: Show that $f(z) = z^{-1}$ does have an antiderivative on the disc $|z+1| < 1$.
  - One way to construct this antiderivative is to find a series expansion for $f(z)$ centered at $z = 1$ and then antidifferentiate it term-by-term inside its radius of convergence.

- Letting $w = z+1$, we want to find an antiderivative for $(w-1)^{-1} = -\dfrac{1}{1-w} = -\sum_{n=0}^{\infty} w^n = -\sum_{n=0}^{\infty}(z+1)^n$, which converges absolutely for $|z+1| < 1$.
- Then integrating termwise yields the antiderivative $F(z) = -\sum_{n=0}^{\infty} \dfrac{(z+1)^{n+1}}{n+1}$, which also converges absolutely for $|z+1| < 1$ and is holomorphic there with derivative $f(z)$.
- Another approach to constructing this antiderivative would be to select a branch of the complex logarithm function with a branch cut that does not intersect the disc $|z+1| < 1$.
- In fact, since the principal complex logarithm $\mathrm{Log}(z)$ is holomorphic on $\mathbb{C}\backslash[0,\infty)$, it provides such an antiderivative.

3.2 Cauchy's Integral Theorem and Formula

- We would like to analyze the existence of antiderivatives on regions more cleanly, since as a practical matter our result that a continuous $f(z)$ has an antiderivative on R if and only if $\int_\gamma f(z)\,dz = 0$ for all contours in R is not practical to verify for specific functions $f(z)$.
  - By necessity, our approach to giving a simpler condition will be somewhat technical, but the short version is that we will want to impose a topological condition on R known as simple connectivity, and we will also require $f(z)$ to be holomorphic.
  - We will begin by motivating the utility of these two conditions using Green's theorem to prove a weaker version of Cauchy's integral theorem, and then we will provide another approach using deformation of contours that also illustrates the necessity of these conditions and gives a stronger statement.
  - We then motivate the second half of our discussion, regarding Cauchy's integral formula, by using our results to greatly simply evaluations of contour integrals of functions defined as convergent Laurent series.
  - Finally, we state and prove the full version of Cauchy's integral formula.

3.2.1 Cauchy's Integral Theorem via Green's Theorem

- First, we recall a basic result from topology:
- Theorem (Jordan Curve Theorem): Suppose $\gamma : [a,b] \to \mathbb{C}$ is a closed, simple, continuous curve. Then its complement $\mathbb{C}\backslash\mathrm{im}(\gamma)$ has exactly two connected components: its unbounded exterior and its bounded interior, each of whose boundary is the curve $\mathrm{im}(\gamma)$.
  - This theorem is quite difficult to prove for general continuous curves, and ultimately the theorem is quite topological in nature.
  - For smooth curves (i.e., curves that are continuously differentiable and where the derivative is everywhere nonzero), one may give a fairly intuitive procedure for deciding whether a point P is in the interior or exterior.
  - Explicitly, by smoothness, there are only finitely many line segments in γ. Now given any point $P \in \mathbb{C}$, draw a ray starting at P in any direction not parallel to any of these line segments, and then count the number of times the ray crosses the curve (which is necessarily finite by smoothness). If the number of crossings is even, then P is in the exterior, while if the number of crossings is odd, then P is in the interior.
  - To make this argument rigorous, one must then show that the number of crossings does not depend on the ray chosen, that the regions of "odd" points and "even" points are connected, and that there is no path from an odd point to an even point. We omit the precise details.
- By the Jordan curve theorem, if γ is a simple closed rectifiable curve, then it encloses an interior region R, which by rectifiability will always be on the same side (either left or right) of γ as one travels around the curve.

- By reversing the direction of travel (i.e., by replacing γ with $-\gamma$, which as we have noted earlier simply negates the associated line integral) if needed, we can also assume that γ has counterclockwise orientation, meaning that R is always on the left of γ.

- Next we recall Green's theorem from multivariable calculus:

- Theorem (Green's Theorem): If γ is a simple closed rectifiable curve oriented counterclockwise in $\mathbb{R}^2$, and R is the region it encloses, then for any continuously differentiable functions $P(x, y)$ and $Q(x, y)$ on R, $\displaystyle\int_\gamma P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx$.
  - Green's theorem is a natural generalization of the fundamental theorem of calculus to planar regions.
  - For illustration we will give the proof of Green's theorem for rectangular regions. One may use this calculation to establish the general proof by observing that both the line integral and the double integral are compatible with "gluing" two regions together along a shared boundary, and then showing that a suitable rectangular approximation of the region R yields close approximations to both the line integral and the double integral.
  - Proof (for rectangles): For a rectangular region $a \le x \le b$, $c \le y \le d$, we have $\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4}$, where γ_1, γ_2, γ_3, and γ_4 are the four sides of the rectangle (with the proper orientation), and the function to be integrated on each curve is $P\,dx + Q\,dy$.
  - Setting up parametrizations shows $\int_{\gamma_1} [P\,dx + Q\,dy] + \int_{\gamma_3} [P\,dx + Q\,dy] = \int_a^b [P(x, c) - P(x, d)]\,dx$, and $\int_{\gamma_2} [P\,dx + Q\,dy] + \int_{\gamma_4} [P\,dx + Q\,dy] = \int_c^d [Q(b, y) - Q(a, y)]\,dy$.
  - For the double integral we have $\iint_R -\dfrac{\partial P}{\partial y}\,dy\,dx = \int_a^b \int_c^d -\dfrac{\partial P}{\partial y}\,dy\,dx = \int_a^b [P(x, c) - P(x, d)]\,dx$, and $\iint_R \dfrac{\partial Q}{\partial x}\,dx\,dy = \int_d^c \int_a^b \dfrac{\partial Q}{\partial x}\,dx\,dy = \int_c^d [Q(b, y) - Q(a, y)]\,dy$.
  - Comparing the expressions shows immediately that $\int_C [P\,dx + Q\,dy] = \iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dy\,dx$, as desired.

- We can convert this statement to a complex analogue by viewing $\mathbb{R}^2$ as $\mathbb{C}$ and allowing $P(x, y)$ and $Q(x, y)$ to be complex-valued.
  - The only additional task is to explain how to interpret line integrals with differentials dx and dy rather than dz.
  - An easy approach is simply to write down Riemann sum definitions for $\int_\gamma f(z)\,dx$ and $\int_\gamma f(z)\,dy$, in the same way we wrote a Riemann sum definition for $\int_\gamma f(z)\,dz$: the only difference is that instead of multiplying the value $f(z_i^*)$ by $z_i - z_{i-1}$, we multiply it by $x_i - x_{i-1} = \mathrm{Re}(z_i) - \mathrm{Re}(z_{i-1})$ or $y_i - y_{i-1} = \mathrm{Im}(z_i) - \mathrm{Im}(z_{i-1})$.
  - By following these definitions through, we see immediately that $\int_\gamma f(z)\,dz = \int_\gamma f(z)[dx + i\,dy] = \int_\gamma f(z)\,dx + i\int_\gamma f(z)\,dy$.
  - Equivalently, and more simply, we have the differential identity $dz = dx + i\,dy$, as one would expect naturally from $z = x + iy$.
  - Furthermore, if we have a parametrization $\gamma(t) = x(t) + iy(t)$ for $a \le t \le b$, then we also have the expected formulas $\int_\gamma f(z)\,dx = \int_a^b f(\gamma(t))\,x'(t)\,dt$ and $\int_\gamma f(z)\,dy = \int_a^b f(\gamma(t))\,y'(t)\,dt$.

- With these definitions in hand, we can pose the complex version of Green's theorem:

- Corollary (Complex Green's Theorem): If γ is a simple closed rectifiable curve oriented counterclockwise in $\mathbb{C}$, and R is the region it encloses, then for any continuously differentiable functions $P(x, y)$ and $Q(x, y)$ on R, we have $\displaystyle\int_\gamma P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx$.
  - The proof follows immediately by breaking P and Q apart into real and imaginary parts and applying the real version of Green's theorem to each.

- Much more interesting is the result when we apply the complex Green's theorem to a line integral using the differential dz and a function $f(z) = u(z) + iv(z)$.
  - Here, we require that $f(z) = u(x + iy)$ be continuously differentiable as a function of x and y on R, which as we have seen is implied by being holomorphic on R.
- Theorem (Cauchy's Integral Theorem, First Version): If γ is a simple closed rectifiable curve oriented counterclockwise in $\mathbb{C}$, and R is the region it encloses, then for any holomorphic function $f(z)$ on R we have $\displaystyle\int_\gamma f(z)\,dz = 0$.
  - Proof: We have $\int_\gamma f(z)\,dz = \int_\gamma f(z)[dx + i\,dy] = \int_\gamma f(z)\,dx + if(z)\,dy$.
  - So if we apply the complex Green's theorem with $P = f(z)$ and $Q = if(z)$, then recalling that $\dfrac{\partial f}{\partial \overline{z}} = \dfrac{1}{2}\left[\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right] = 0$ because f is holomorphic, we obtain $\int_\gamma f(z)\,dx + if(z)\,dy = \iint_R \left(i\dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial y}\right) dy\,dx = 2i \iint_R \left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right) dy\,dx = 2i \iint_R \dfrac{\partial f}{\partial \overline{z}}\,dy\,dx = 2i \iint_R 0\,dy\,dx = 0$.
  - Therefore, $\int_\gamma f(z)\,dz = 0$, as claimed.
- We will remark, as a key point, that although we are only computing the line integral around the boundary γ of R, in fact we do require $f(z)$ to be holomorphic on all of the interior region R for the theorem to hold.
  - Clearly this fact is required in the proof, since we need to be able to evaluate the double integral $\iint_R \dfrac{\partial f}{\partial \overline{z}}\,dy\,dx$ on R.
  - However, as the example of $f(z) = 1/z$ with γ the counterclockwise boundary of the unit circle shows, we cannot even drop the assumption of holomorphicity at a single point in R without making the theorem false, since $\int_\gamma 1/z\,dz = 2\pi i$ is nonzero yet $f(z) = 1/z$ only fails to be holomorphic at the single point $z = 0$.
  - If we switch focus from the curve γ to the region R, we can clearly drop the requirement that γ run counterclockwise, since reversing the direction will simply scale the integral by -1, it will still be zero. So the regions R on which the theorem applies are obtained by filling in the interior of a simple closed rectifiable curve.

3.2.2 Cauchy's Integral Theorem via Deformation of Contours

- Intuitively, we can describe the regions R in our statement of Cauchy's integral theorem above as the ones that have no "holes" in them. We would like to formalize this notion, which we do using homotopy.
  - For convenience, by rescaling the parametrization, we will assume that all of our curves are parametrized as $\gamma : [0,1] \to \mathbb{C}$.
- Definition: Suppose that $\gamma_0 : [0,1] \to R$ and $\gamma_1 : [0,1] \to R$ are continuous closed curves in a region R. We say that a function $h : [0,1] \times [0,1] \to R$ is a homotopy in R from γ_0 to γ_1 when h is continuous, $h(s,0) = h(s,1)$ for all s, and $h(0,t) = \gamma_0(t)$ and $h(1,t) = \gamma_1(t)$ for all $t \in [0,1]$. If there exists some homotopy in R from one curve to another, we say the curves are homotopic in R.
  - A homotopy is a continuous deformation of γ_0 into γ_1: the idea is, roughly, that the homotopy describes how to "slide" γ_0 through the plane so that it becomes γ_1. Explicitly, we describe this deformation via the intermediate curves $\gamma_s : [0,1] \to \mathbb{C}$ where $\gamma_s(t) = h(s,t)$ for each $0 \le s \le 1$. The condition $h(s,0) = h(s,1)$ is then simply that each intermediate curve γ_s is also closed.
  - It is not hard to verify that being homotopic in R is an equivalence relation on continuous curves in R. Additionally, whether or not two curves are homotopic will depend on the region R in addition to the curves.

- Example: If $R = \mathbb{C}$ and $\gamma_1 : [0,1] \to \mathbb{C}$ is any continuous closed curve, then γ_1 is homotopic to the "trivial curve" $\gamma_0(t) = 0$ for all $0 \le t \le 1$, via the homotopy $h(s,t) = s\gamma_1(t)$. We can see that $h(s,t)$ is continuous (since it is the product of two continuous functions), that $h(s,0) = s\gamma_1(0) = s\gamma_1(1) = h(s,1)$, and that $h(0,t) = 0 = \gamma_0(t)$ and $h(1,t) = \gamma_1(t)$, so it is indeed a homotopy.
- As a consequence, since homotopy is an equivalence relation, we see that any two continuous closed curves are homotopic in $\mathbb{C}$.
- It may therefore seem that our definition of homotopic curves is rather trivial, but that is only because there are no obstacles to deforming curves in $\mathbb{C}$. If we remove points from $\mathbb{C}$, we can create obstacles that prevent two curves from being homotopic.
- Example: If $R = \mathbb{C}\backslash\{0\}$ and γ_1 is the curve winding once counterclockwise around the unit circle, then γ_1 is not homotopic to the trivial curve $\gamma_0(t) = 1$ for all $0 \le t \le 1$. This result is not so easy to prove rigorously[2], but the point is that since γ_0 contains 0 in its interior, any continuous deformation into another curve γ_1 that does not contain 0 in its interior must have an intermediate curve γ_s in R that passes through 0, which is not allowed since R does not contain 0.

* For nice regions, such as the interior R of a continuous curve (per the Jordan curve theorem), every continuous closed curve in R is homotopic to a point (i.e., a constant curve). These are the regions we wish to discuss:
* Definition: If R is a region in $\mathbb{C}$, we say R is simply connected if R is connected and every continuous curve $\gamma : [0,1] \to R$ is homotopic to a point.
  - Intuitively, the idea is that if we draw any curve in R, we may simply shrink the curve (without leaving R) until it is a single point. This prevents the region from having any "holes", since a curve drawn around such a hole could not be shrunk to a point.
  - As is often shown as part of the Jordan curve theorem, if R is the interior (or exterior) of a region whose boundary is a continuous closed curve, then R is simply connected. In particular, every region to which we can apply Green's theorem (and thus the Cauchy integral theorem we proved using Green's theorem) is simply connected.
* We will now give another proof of Cauchy's integral theorem that relies on deformation of contours. The main engine for this approach is the following very useful fact:
* Theorem (Deformation of Contours): Suppose R is a region such that γ_1 and γ_2 are closed rectifiable curves in R that are homotopic in R and $f(z)$ is holomorphic on R. Then $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$.
  - For arbitrary homotopies this result requires stronger techniques using approximations. We will give a proof in the simpler case where the homotopy function $h(s,t)$ is twice continuously differentiable in s and t and $f'(z)$ is continuous.
  - We recall Leibniz's rule for differentiation under the integral sign, which allows us to change the order of an integral and a partial derivative, analogously to how Clairaut's theorem allows us to interchange the order of partial derivatives (when those associated partial derivatives are continuous) and Fubini's theorem allows us to interchange the order of multiple integrals (when the integrand is absolutely integrable).
  - Explicitly, Leibniz's rule says that if $f(x,t)$ and its x-partial $f_x(x,t)$ are continuous in x and t, then $\dfrac{d}{dx}\left[\int_a^b f(x,t)\,dt\right] = \int_a^b f_x(x,t)\,dt$.
  - Proof (Special Case): Suppose that $\gamma_1 : [0,1] \to R$ and $\gamma_2 : [0,1] \to R$ are closed rectifiable curves in R and $h(s,t)$ is a twice continuously differentiable homotopy from γ_1 to γ_2.
  - Consider the function $I(s) = \int_0^1 f(h(s,t))\,h_t(s,t)\,dt$: we will show $I'(s) = 0$.
  - Since f, h, and h_t are all continuously differentiable in all variables, so is the product $f(h(s,t))\,h_t(s,t)$.

[2] A direct approach is as follows: first, the region $\mathbb{C}\backslash\{0\}$ can be continuously deformed onto the onto the unit circle via the homotopy $h(s, re^{i\theta}) = (1 - r\cot(s\pi/2))e^{i\theta}$, so it suffices to show that the path winding once around the unit circle is not homotopic to a constant map when the region R is just the unit circle itself. To show this consider the set S of $s \in [0,1]$ such that $-1 \notin \gamma_s$. Then since $h(s,t)$ is continuous, S is closed and also S^c must be closed (these follow by using $\epsilon/2$ arguments in either direction). But since $0 \in S$ and $1 \in S^c$ this would contradict the connectedness of the interval $[0,1]$, which is a contradiction. Another less ad hoc approach is to use properties of covering spaces, or equivalently to show that the path winding once around the unit circle is a generator of the fundamental group $\pi_1(S^1) \cong \mathbb{Z}$ of the circle S^1 while the constant path corresponds to the identity element.

* Thus, by Leibniz's differentiation rule, the product and chain rules, Clairaut's theorem applied to obtain $h_{st} = h_{ts}$, and the fundamental theorem of line integrals, we have

$$\begin{aligned} I'(s) = \frac{d}{ds}\left[\int_0^1 f(h(s,t))\, h_t(s,t)\, dt\right] &= \int_0^1 \frac{\partial}{\partial s}\left[f(h(s,t))\, h_t(s,t)\right] dt \\ &= \int_0^1 \left[f'(h(s,t))\, h_s(s,t) h_t(s,t) + f(h(s,t))\, h_{st}(s,t)\right] dt \\ &= \int_0^1 \left[f'(h(s,t))\, h_t(s,t) h_s(s,t) + f(h(s,t))\, h_{ts}(s,t)\right] dt \\ &= \int_0^1 \frac{\partial}{\partial t}[f(h(s,t)) h_s(s,t)]\, dt \\ &= f(h(s,1)) h_s(s,1) - f(h(s,0)) h_s(s,0) \end{aligned}$$

  and now finally this last expression is zero because the function $\gamma_s : [0,1] \to R$ with $\gamma_s(t) = h(s,t)$ is a closed curve, so $h(s,1) = h(s,0)$ and $h_s(s,1) = h_s(s,0)$.
* Thus $I'(s) = 0$ for $0 < s < 1$, so I is constant on $[0,1]$ hence in particular we have $I(0) = I(1)$.
* Finally, since $\int_{\gamma_1} f(z)\, dz = \int_0^1 f(\gamma_1(t))\, \gamma_1'(t)\, dt = \int_0^1 f(h(0,t))\, h_t(0,t)\, dt = I(0)$ and similarly $\int_{\gamma_2} f(z)\, dz = \int_0^1 f(\gamma_2(t))\, \gamma_2'(t)\, dt = \int_0^1 f(h(1,t))\, h_t(1,t)\, dt = I(1)$, we obtain $\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$ as claimed.

- Using this result we can evaluate many contour integrals by transforming the shape into one where the integral is easier to calculate via a parametrization.
  * As an actual practical matter, the easiest closed curve to parametrize is a circle, since we do not need to break it into several components like we would for a triangle or rectangle.
- Example: Evaluate $\int_\gamma z^{-1}\, dz$ where $\gamma(t) = 2\cos(t) + 4i\sin(t)$ for $0 \le t \le 2\pi$.
  * We see that $f(z) = z^{-1}$ is holomorphic on $R = \mathbb{C}\backslash\{0\}$, so we may freely deform the contour as long as we avoid the point $z = 0$.
  * Since the contour is simply the counterclockwise ellipse $x^2/4 + y^2/16 = 1$, we may simply stretch it continuously until it becomes the counterclockwise unit circle $x^2 + y^2 = 1$, without causing the curve to pass through $z = 0$.
  * Thus, if γ_2 is the boundary of the unit circle, we see $\int_\gamma z^{-1}\, dz = \int_{\gamma_2} z^{-1}\, dz = \boxed{2\pi i}$ as we have previously calculated.
- Example: Evaluate $\int_\gamma \frac{z}{z^2 - 2z + 1}\, dz$ where γ is the counterclockwise boundary of the square with vertices $\pm 3 \pm 3i$.
  * We see that $f(z)$ is holomorphic on $R = \mathbb{C}\backslash\{1\}$ so we may freely deform the contour as long as we avoid $z = 1$.
  * A quick sketch shows that we can continuously shrink the contour until it becomes the circle $|z - 1| = 2$ without passing through $z = 1$.
  * We can parametrize this circle as $\gamma(t) = 1 + 2e^{it}$ for $0 \le t \le 2\pi$, with $\gamma'(t) = 2ie^{it}$. Then the integrand is $f(z) = \frac{z}{z^2 - 2z + 1} = \frac{1 + 2e^{it}}{(2e^{it})^2} = \frac{1}{4}(e^{-2it} + 2e^{-it})$, so $\int_\gamma \frac{z}{z^2 - 2z + 1}\, dz = \int_0^{2\pi} \frac{1}{4}(e^{-2it} + 2e^{-it})(2ie^{it})\, dt = \int_0^{2\pi} \frac{i}{2}(e^{-it} + 2)\, dt = \boxed{2\pi i}$.
- In the situation where R is simply connected, every continuous closed curve is homotopic to a point. Applying this fact to our deformation of contours result immediately yields Cauchy's integral theorem (again), but with a less restrictive hypothesis:
- Corollary (Cauchy's Integral Theorem, Second Version): If γ is a closed rectifiable curve and R is a simply connected region, then for any holomorphic function $f(z)$ on R we have $\int_\gamma f(z)\, dz = 0$.

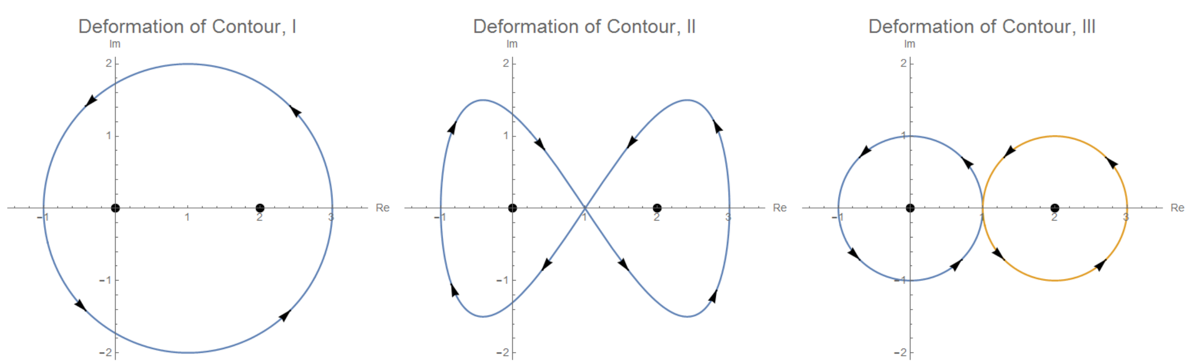
- Proof: Suppose γ is a closed rectifiable curve and R is a simply connected region. Then by definition, γ is homotopic to a point: namely, a constant curve γ_0.
- By the deformation of contours theorem, we then have $\int_\gamma f(z)\,dz = \int_{\gamma_0} f(z)\,dz$ and this last integral is zero since the contour is trivial. (Indeed, all of its associated Riemann sums are even zero.)

- As a consequence of this stronger version of Cauchy's integral theorem, we also see that every holomorphic function on a simply-connected region has an antiderivative:

- Corollary (Existence of Antiderivatives, II): Suppose R is a simply connected open region and $f(z)$ is holomorphic on R. Then there exists a holomorphic function $F(z)$ such that $F'(z) = f(z)$ on R.
  - Proof: By Cauchy's integral theorem, we have $\displaystyle\int_\gamma f(z)\,dz = 0$ for all closed rectifiable curves in R.
  - Since $f(z)$ is continuous, by our previous result on existence of antiderivatives, for any fixed $a \in R$ and rectifiable path γ_0 from a to z_0 (which exists since R is connected), the function $F(z_0) = \int_{\gamma_0} f(z)\,dz$ has $F'(z) = f(z)$ on R.

- By deforming contours in more complicated ways, and also by simplifying integrands, we can often reduce difficult contour integrals to simpler ones that can be evaluated directly.

- Example: Evaluate $\int_\gamma \dfrac{4z-4}{z^2-2z}\,dz$ where $\gamma(t) = 1 + 2e^{it}$ for $0 \le t \le 2\pi$.
  - We could simply plug in the parametrization, but the resulting rational function in $\sin t$ and $\cos t$ is rather messy and unpleasant to evaluate (though of course, it is possible).
  - Additionally, the integrand is holomorphic on $R = \mathbb{C}\backslash\{0,2\}$ so we may freely deform the contour as long as we avoid both of those points.
  - Now imagine "pinching" the circle in the middle along $\mathrm{Re}(z) = 1$ until it reaches a "figure eight" shape, and then round out the sides to form two circles:



  - Since contour integrals are additive on curves, it suffices to calculate the integral on the curve γ_1 (the counterclockwise unit circle) and γ_2 (the counterclockwise circle $|z-2| = 1$).
  - Furthermore, from partial fraction decomposition we have $\dfrac{4z-4}{z^2-4z} = \dfrac{2}{z} + \dfrac{2}{z-2}$, so we need only calculate $\int_{\gamma_1} \dfrac{1}{z}\,dz$, $\int_{\gamma_1} \dfrac{1}{z-2}\,dz$, $\int_{\gamma_2} \dfrac{1}{z}\,dz$, and $\int_{\gamma_2} \dfrac{1}{z-2}$.
  - Using our previous calculations we have $\int_{\gamma_1} \dfrac{1}{z}\,dz = 2\pi i$ and also $\int_{\gamma_2} \dfrac{1}{z-2}\,dz = 2\pi i$ (since this integral is the same as the first one after substituting $w = z-2$).
  - Furthermore, since $\dfrac{1}{z}$ is holomorphic on the region enclosed by γ_2, and $\dfrac{1}{z-2}$ is holomorphic on the region enclosed by γ_1, we also have $\int_{\gamma_1} \dfrac{1}{z-2}\,dz = \int_{\gamma_2} \dfrac{1}{z}\,dz = 0$.
  - Putting all of this together shows that $\int_\gamma \dfrac{4z-4}{z^2-2z}\,dz = \int_{\gamma_1} \dfrac{2}{z}\,dz + \int_{\gamma_1} \dfrac{2}{z-2}\,dz + \int_{\gamma_2} \dfrac{2}{z}\,dz + \int_{\gamma_2} \dfrac{2}{z-2}\,dz = 2(2\pi i) + 2(2\pi i) = \boxed{8\pi i}$.

3.2.3 Integration of Power Series and Laurent Series

- Our next goal is to discuss integration of functions defined by convergent Laurent series. We might hope that we could simply integrate such series term by term, and that is indeed the case inside their (punctured) disc of convergence:

- Proposition (Integration and Laurent Series): Suppose γ is a rectifiable curve in the region R. Then the following hold:

  1. If the continuous functions $f_n(z)$ converge uniformly to $f(z)$ on R, then $\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz$.
     - Similar to many other results involving uniform convergence, this result allows us to interchange the integral and the limit.
     - We will also use a fact about lengths of curves: formally, the length of γ is defined as $L = \int_a^b |\gamma'(t)|\,dt$. Since γ' is continuous except at a finite number of points, the length of γ is finite.
     - Proof: As we have shown previously, the uniform limit of continuous functions is continuous, so $f(z)$ is continuous and hence integrable.
     - Now let $\epsilon > 0$ and suppose that $\gamma : [a,b] \to \mathbb{C}$ has length L. By uniform convergence there exists N such that $|f_n(z) - f(z)| < \dfrac{\epsilon}{L}$ for all $z \in R$ and all $n \geq N$.
     - Then $\left|\int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz\right| = \left|\int_a^b [f_n(\gamma(t)) - f(\gamma(t))]\gamma'(t)\,dt\right| \leq \int_a^b |f_n(\gamma(t)) - f(\gamma(t))|\,|\gamma'(t)|\,dt < \int_a^b \dfrac{\epsilon}{L} |\gamma'(t)|\,dt = \dfrac{\epsilon}{L} \int_a^b |\gamma'(t)|\,dt = \epsilon$ by the triangle inequality.
     - Thus by the definition of limit, since $\left|\int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz\right| < \epsilon$ for all $n \geq N$, we have $\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz$ as claimed.
  2. If the Laurent series $f(z) = \sum_{n=-k}^{\infty} a_n (z-a)^n$ has positive radius of convergence R, and γ is a rectifiable closed curve contained entirely within the punctured disc $0 < |z-a| < R$, then $\int_\gamma f(z)\,dz = \sum_{n=-k}^{\infty} a_n \int_\gamma (z-a)^n\,dz$.
     - Proof: Since $\gamma : [0,1] \to \mathbb{C}$ is continuous, the function $g(t) = |\gamma(t) - a|$ is continuous on $[0,1]$ hence by the extreme value theorem it attains its minimum and maximum values.
     - Since γ cannot pass through 0, the minimum must be some positive value $r_{\min} > 0$, and since γ is contained within the disc $|z-a| < R$ the maximum must be some value $r_{\max} < R$.
     - On the closed disc $r_{\min} \leq |z-a| \leq r_{\max}$, as we have shown, the partial sums $f_d(z) = \sum_{n=-k}^{d} a_d (z-a)^d$ are continuous and converge uniformly to f.
     - Thus by (1), the integral of their limit $f(z)$ is equal to the limit $\lim_{d\to\infty} \int_\gamma [\sum_{n=-k}^{d} a_n (z-a)^n]\,dz$, which equals $\sum_{n=-k}^{\infty} a_n \int_\gamma (z-a)^n\,dz$ since the sum is finite.
  3. If γ is any rectifiable closed curve in the punctured disc $0 < |z-a| < R$ and n is any integer not equal to -1, then $\int_\gamma (z-a)^n\,dz = 0$. If $n \geq 0$ then the result also holds if γ passes through $z = a$.
     - Proof: For $n \neq -1$ we see that $(z-a)^n$ has an antiderivative $\dfrac{(z-a)^{n+1}}{n+1}$ defined for all $z \neq a$ (if $n \geq 0$, of course, this function is also defined at $z = a$).
     - But then $\int_\gamma (z-a)^n\,dz = 0$ by the fundamental theorem of line integrals, since γ is closed.
  4. If the Laurent series $f(z) = \sum_{n=-k}^{\infty} a_n (z-a)^n$ has positive radius of convergence R, and γ is a rectifiable closed curve contained entirely within the punctured disc $0 < |z-a| < R$, then $\int_\gamma f(z)\,dz = a_{-1} \int_\gamma (z-a)^{-1}\,dz$.
     - Proof: This follows immediately by plugging (3) into (2).

- Quite interestingly, we see that the value of the integral of a function defined by a convergent Laurent series depends only on the coefficient a_{-1} of the series along with the value of the integral of $(z-a)^{-1}$ around the contour.
  - As noted previously, this behavior is intimately linked to the fact that z^{-1} has no antiderivative on a punctured disc of positive radius around 0.

- By deforming contours appropriately, we can evaluate any contour integral of this form.
- The main idea is to observe that if γ is the circle $|z-a| = r$ traversed once counterclockwise, then by using the parametrization $z = a + re^{it}$ for $0 \le t \le 2\pi$, we see $\int_\gamma (z-a)^{-1}\,dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$, regardless of the value of r.
- Thus, we may complete our evaluation of $\int_\gamma f(z)\,dz$ by deforming the contour into one that traverses a circle centered at $z = a$ some number of times.
- We will content ourselves only to give some motivating examples at the moment, since we aim ultimately to generalize this discussion to all holomorphic functions, rather than just those defined locally by a convergent Laurent series.

- Example: Find $\int_\gamma \frac{e^z}{z^4}\,dz$ where $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$.
  - As a Laurent series centered at $z = 0$, we have $\frac{e^z}{z^4} = z^{-4} + z^{-3} + \frac{z^{-2}}{2!} + \frac{z^{-1}}{3!} + \frac{1}{4!} + \cdots$, which converges for all $z \ne 0$. Note also that γ traverses the unit circle once counterclockwise.
  - Thus, by our discussion above, we see that $\int_\gamma \frac{e^z}{z^3}\,dz = \frac{1}{3!}\int_\gamma z^{-1}\,dz = \frac{1}{3!}(2\pi i) = \boxed{\frac{1}{3}\pi i}$.

- Example: Find $\int_\gamma \csc(2z)\,dz$ where $\gamma(t)$ is the counterclockwise boundary of the square with vertices ± 1, $\pm i$.
  - As a Laurent series we have $\csc(2z) = \frac{1}{\sin(2z)} = \frac{1}{(2z) - (2z)^3/3! + \cdots} = \frac{1}{2}z^{-1} + \frac{2}{3!}z + \cdots$, which converges for all $z \ne 0$ since the series for $\sin(2z)$ converges for all z.
  - Therefore, we have $\int_\gamma \csc z\,dz = \frac{1}{2}\int_\gamma z^{-1}\,dz$.
  - It is easy to see from the description that we may continuously deform γ into the counterclockwise boundary γ_2 of the unit circle.
  - Thus, we see $\int_\gamma \csc(2z)\,dz = \frac{1}{2}\int_\gamma z^{-1}\,dz = \frac{1}{2}\int_{\gamma_2} z^{-1}\,dz = \frac{1}{2}(2\pi i) = \boxed{\pi i}$.

- Example: Find $\int_\gamma \frac{\cos z}{z^4}\,dz$ where $\gamma(t)$ is the closed polygonal path with successive vertices 1, $-1+i$, $-1-i$, $5+i$, $-7+3i$, $-8+4i$, $4-11i$, and then finally returning to 1.
  - As a Laurent series we have $\frac{\cos z}{z^4} = z^{-4} - \frac{1}{2!}z^{-2} + \frac{1}{4!} - \cdots$, which converges for $z \ne 0$.
  - Since the coefficient of z^{-1} is zero we see that the integral is also $\boxed{0}$, without even needing to evaluate $\int_\gamma z^{-1}\,dz$.

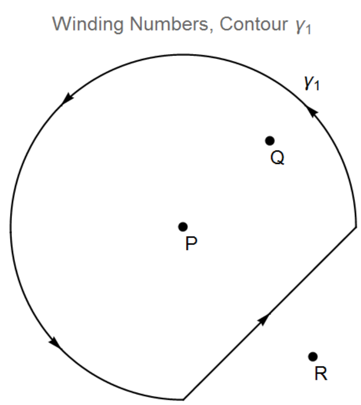
- Example: Find $\int_\gamma \frac{\cos z}{z^3}\,dz$ where $\gamma(t)$ is the closed polygonal path with successive vertices 1, $-1+i$, $-1-i$, $5+i$, $-7+3i$, $-8+4i$, $4-11i$, and then finally returning to 1.
  - As a Laurent series we have $\frac{\cos z}{z^3} = z^{-3} - \frac{1}{2!}z^{-1} + \frac{1}{4!}z - \cdots$, which converges for $z \ne 0$.
  - Now we do need to evaluate $\int_\gamma z^{-1}\,dz$. By drawing the contour, we can see that crosses itself, and winds around 0 twice. By separating the contour into the two separate pieces and then deforming each of the resulting simple loops into the unit circle, we can see that $\int_\gamma z^{-1}\,dz = 2\int_{\gamma_2} z^{-1}\,dz = 4\pi i$.
  - Hence by our results we have $\int_\gamma \frac{\cos z}{z^3}\,dz = -\frac{1}{2!}\int_\gamma z^{-1}\,dz = \boxed{-2\pi i}$.

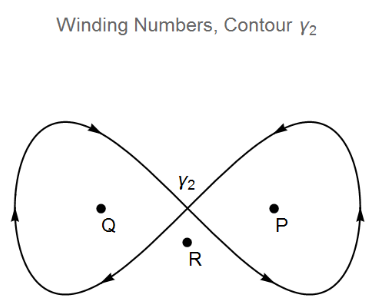
3.2.4 Winding Numbers

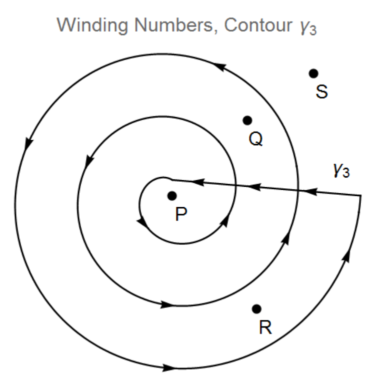
- Our calculations with integrating Laurent series indicated that for any $f(z)$ described by a convergent Laurent series centered at $z = z_0$ and any closed γ that lies within its disc of convergence, we can reduce the calculation of $\int_\gamma f(z)\,dz$ to the calculation of $\int_\gamma \frac{1}{z - z_0}\,dz$.
  - Our goal is to show that this same sort of reduction can be made for arbitrary holomorphic functions.
- First, we will study the integrals $\int_\gamma \frac{1}{z - z_0}\,dz$ in more detail, where γ is any closed contour not passing through z_0.
  - As we have shown several times, if γ is any circle of positive radius centered at $z = z_0$ traversed once counterclockwise, the integral $\int_\gamma \frac{1}{z - z_0}\,dz$ has value $2\pi i$. More generally, if γ traverses the circle n times counterclockwise (where n could be 0 or negative), then $\int_\gamma \frac{1}{z - z_0}\,dz = n \cdot 2\pi i$.
  - Furthermore, since the value of the contour integral is invariant under continuous deformation of the contour, if we have any contour that is homotopic in $\mathbb{C}\backslash\{0\}$ to a circular path that traverses the unit circle n times counterclockwise, then $\int_\gamma \frac{1}{z - z_0}\,dz = n \cdot 2\pi i$.
  - Intuitively, the antiderivative of $\frac{1}{z - z_0}$, in its most general possible sense, should be the complex logarithm $\log(z - z_0)$: but this is a multivalued function, so it does not make sense, strictly speaking, to use it to evaluate the integral using the fundamental theorem of line integrals.
  - Nonetheless, formally, if $\gamma : [a, b] \to \mathbb{C}$, we would want to write something along the lines of $\int_\gamma \frac{1}{z - z_0}\,dz = \log(\gamma(b) - z_0) - \log(\gamma(a) - z_0)$. Then since $\gamma(a) = \gamma(b)$, if we allow each of the logarithms to take any of their possible values, the possible values of the difference are the integer multiples of $2\pi i$.
  - This calculation, although not rigorous, suggests that we should expect the actual value of this integral $\int_\gamma \frac{1}{z - z_0}\,dz$ to be an integer multiple of $2\pi i$.
  - In fact, if we instead work with a specific branch of the logarithm (e.g., the principal logarithm $\mathrm{Log}(z)$), we can pin down more precisely what this multiple of $2\pi i$ represents.
  - Specifically, the branch cut of the logarithm will separate γ into some finite number of pieces. Then we can sum the contributions made to $\int_\gamma \frac{1}{z - z_0}\,dz$ on all of the pieces of γ separated by the branch cut, and then keep track of the difference in the value of $\int_a^b \frac{1}{z - z_0}\,dz$ and the value $\mathrm{Log}(b - z_0) - \mathrm{Log}(a - z_0)$ each time the curve γ crosses over the branch cut. Because of the nature of the discontinuity, this difference will be $+2\pi i$ if γ crosses the branch cut upwards and $-2\pi i$ if γ crosses the branch cut downwards.
  - Tallying up over all of the pieces shows that the integral differs from 0 by an integer multiple of $2\pi i$ given by the total number of "upward crossings" minus the number of "downward crossings" of γ across the branch cut.
  - In this way, we see that the value $\frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\,dz$, roughly speaking, is counting the number of times the contour γ "winds around" the point $z = z_0$ in the counterclockwise direction.
  - We now make this notion precise:
- <u>Proposition</u> (Winding Numbers): Suppose γ is any closed contour not passing through $z_0 \in \mathbb{C}$. Then the value $\frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\,dz$ is always an integer; we refer to this integer as the <u>winding number</u> of γ around z_0 and denote it as $W_\gamma(z_0)$.
  - <u>Proof</u>: By deforming the contour appropriately we may assume that γ is continuously differentiable (simply smooth any nondifferentiable points, making sure to avoid z_0 while doing so). So suppose $\gamma : [a, b] \to \mathbb{C}$ is continuously differentiable: then $\int_\gamma \frac{1}{z - z_0}\,dz = \int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0}\,dt$.

- Now define the function $F(x) = \int_a^x \dfrac{\gamma'(t)}{\gamma(t) - z_0}\,dt$ for $a \le x \le b$. By the fundamental theorem of calculus, we have $F'(x) = \dfrac{\gamma'(x)}{\gamma(x) - z_0}$.
- Consider the function $g(x) = e^{F(x)}(\gamma(x) - z_0)^{-1}$: its derivative is $\dfrac{d}{dx}\left[e^{F(x)}(\gamma(x) - z_0)^{-1}\right] = e^{F(x)}F'(x)(\gamma(x) - z_0)^{-1} - e^{F(x)}(\gamma(x) - z_0)^{-2}\gamma'(x) = 0$ using the formula for $F'(x)$ above.
- Therefore $g'(x)$ is identically zero for $a \le x \le b$, so $g(x)$ is constant on $[a, b]$. In particular, $g(b) = g(a)$, so that $e^{F(b)}[\gamma(b) - z_0]^{-1} = e^{F(a)}[\gamma(a) - z_0]^{-1}$.
- But since $\gamma(b) = \gamma(a)$ and $F(a) = \int_a^a \dfrac{\gamma'(t)}{\gamma(t) - z_0}\,dt = 0$, we immediately see that $e^{F(b)}[\gamma(b) - z_0]^{-1} = e^{F(a)}[\gamma(a) - z_0]^{-1}$ implies $e^{F(b)} = 1$, and so by properties of exponentials this means $F(b) = \int_a^b \dfrac{\gamma'(t)}{\gamma(t) - z_0}\,dt = \int_\gamma \dfrac{1}{z - z_0}\,dz$ is an integer multiple of $2\pi i$, as claimed.
- Remark: The motivation for examining the function $e^{F(x)}(\gamma(x) - z_0)^{-1}$ is that $F(x)$ should be the complex logarithm $\log(\gamma(x) - z_0)$: working with $e^{F(x)}$ allows us to avoid the issues with having a multivalued logarithm, and $e^{F(x)}$ should simply be $\gamma(x) - z_0$, so it is natural to try to establish the desired result by showing $e^{F(x)}(\gamma(x) - z_0)^{-1}$ has zero derivative since this function should just be identically 1.

- Intuitively, the winding number of a contour around a point is the net number of times the contour winds counterclockwise around that point. We can calculate winding numbers visually:







- For the contour γ_1 drawn above, the winding number around P is 1, the winding number around Q is 1, and the winding number around R is 0.
- As a consequence, we therefore have $\int_{\gamma_1} \dfrac{1}{z - P}\,dz = 2\pi i$, $\int_{\gamma_1} \dfrac{1}{z - Q}\,dz = 2\pi i$, and $\int_{\gamma_1} \dfrac{1}{z - R}\,dz = 0$.
- For the contour γ_2 drawn above, the winding number around P is 1, the winding number around Q is -1, and the winding number around R is 0.
- As a consequence, we therefore have $\int_{\gamma_2} \dfrac{1}{z - P}\,dz = 2\pi i$, $\int_{\gamma_2} \dfrac{1}{z - Q}\,dz = -2\pi i$, and $\int_{\gamma_2} \dfrac{1}{z - R}\,dz = 0$.
- For the contour γ_3 drawn above, the winding number around P is 3, the winding number around Q is 2, the winding number around R is 1, and the winding number around S is 0.
- As a consequence, we therefore have $\int_{\gamma_3} \dfrac{1}{z - P}\,dz = 6\pi i$, $\int_{\gamma_3} \dfrac{1}{z - Q}\,dz = 4\pi i$, $\int_{\gamma_3} \dfrac{1}{z - R}\,dz = 2\pi i$, and $\int_{\gamma_3} \dfrac{1}{z - S}\,dz = 0$.

- We would also expect, based on geometric intuition, that winding numbers should not change as we vary the point z_0, as long as we do not cross the contour γ, which we can make precise as follows:
- Proposition (Properties of Winding Numbers): Suppose γ is a closed contour in $\mathbb{C}$.

1. For z_0 not lying on γ, the function $W_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz$ is continuous in z_0.

   - Proof: We want $\lim_{z_0\to a} W_\gamma(z_0) = W_\gamma(a)$, which is equivalent to $\lim_{z_0\to a}\int_\gamma \left[\frac{1}{z-z_0} - \frac{1}{z-a}\right] dz = 0$.
   - Suppose $\gamma : [0,1] \to \mathbb{C}$. Then since the function $g(t) = |\gamma(t) - a|$ is continuous on the closed interval $[0,1]$, it attains its minimum value M. Note that $M > 0$ because a does not lie on γ.
   - For $|z_0 - a| < M/2$ by the triangle inequality we see that $|\gamma(t) - z_0| > M/2$ for all $t \in [0,1]$.
   - Therefore, for such z_0, we have $\left|\frac{1}{z-z_0} - \frac{1}{z-a}\right| = \left|\frac{a-z_0}{(z-z_0)(z-a)}\right| \le \frac{|z_0-a|}{(M/2)M} = \frac{2}{M^2}|z_0-a|$.
   - Now let $\epsilon > 0$ and suppose γ has length s. Then for $\delta = \min(\frac{M}{2}, \frac{M^2}{2s}\epsilon)$, for $|z_0 - a| < \delta$ we see that $|z_0 - a| < M/2$ so by the calculations above we have $\left|\int_\gamma \left[\frac{1}{z-z_0} - \frac{1}{z-a}\right] dz\right| \le s\cdot\max_{z\in\gamma}\left|\frac{1}{z-z_0} - \frac{1}{z-a}\right| \le s\cdot\frac{2}{M^2}|z_0-a| < s\cdot\frac{2}{M^2}\cdot\frac{M^2}{2s}\epsilon = \epsilon$, as required.
   - Therefore $\lim_{z_0\to a}\int_\gamma \left[\frac{1}{z-z_0} - \frac{1}{z-a}\right] dz = 0$ and so $\lim_{z_0\to a} W_\gamma(z_0) = W_\gamma(a)$ as claimed.

2. If S is a connected region not intersecting the image of γ, then the function $W_\gamma(z)$ is constant on S.
   - As an immediate consequence, if γ is a Jordan curve, then the winding number $W_\gamma(z)$ is constant on the interior of γ.
   - Proof: As we showed in (1), $W_\gamma(z)$ is a continuous function of z. By our proposition above on winding numbers, $W_\gamma(z)$ is integer-valued.
   - Now let a, b be any points in S: since S is connected consider any continuous path in S that joins them. The winding number $W_\gamma(z)$ is then a continuous integer-valued function along this path, but such a function must be constant by the intermediate value theorem (if it took distinct integer values n_1 and n_2, then it would also take every other real value between n_1 and n_2).
   - Therefore $W_\gamma(a) = W_\gamma(b)$ for any $a, b \in S$, so $W_\gamma(z)$ is constant on S.

3. If S is a connected unbounded region not intersecting the image of γ, then $W_\gamma(z)$ is identically zero on S.
   - As an immediate consequence, if γ is a Jordan curve, then the winding number $W_\gamma(z)$ is identically zero on the exterior of γ.
   - Proof: If S is unbounded it contains z_0 with arbitrarily large absolute values.
   - Suppose $\gamma : [0,1] \to \mathbb{C}$ has length s. Since $g(t) = |\gamma(t)|$ is continuous on $[0,1]$ it attains its maximum value M.
   - Then by the triangle inequality we have $\left|\frac{1}{z-z_0}\right| \le \frac{1}{|z_0| - M}$, yielding the simple estimate $|W_\gamma(z_0)| \le \frac{s}{2\pi i}\cdot\frac{1}{|z_0| - M}$.
   - As $|z_0| \to \infty$ this quantity tends to zero, so since it is integer-valued, we must have $W_\gamma(z_0) = 0$ for sufficiently large $|z_0|$.
   - But then by (2), since $W_\gamma(z)$ is constant on S and $W_\gamma(z_0) = 0$ for some $z_0 \in S$, we have $W_\gamma(z) = 0$ for all $z \in S$.

3.2.5 Cauchy's Integral Formula

- As we have seen, if $f(z)$ is holomorphic on a simply connected region R, then $\int_\gamma f(z)\,dz = 0$.
  - We have also seen that for a function represented by a Laurent series $f(z) = \sum_{n=-k}^{\infty} a_n(z-a)^n$ with positive radius of convergence R, and γ is a rectifiable closed curve contained entirely within the punctured disc $0 < |z-a| < R$, then $\int_\gamma f(z)\,dz = a_{-1}\int_\gamma (z-a)^{-1}\,dz$.
  - Since the given integral is simply $2\pi i$ times the winding number $W_\gamma(a)$ we see that the value of the integral $\int_\gamma f(z)\,dz$ is given by the comparatively simple formula $2\pi i a_{-1} W_\gamma(a)$.

- In particular, for any contour winding once counterclockwise around $z = a$, we see $\int_\gamma f(z)\,dz = 2\pi i a_{-1}$, or, equivalently, $a_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)\,dz$.
- In other words, we can calculate the coefficient of $(z-a)^{-1}$ in the Laurent series expansion of $f(z)$ by evaluating the integral $\frac{1}{2\pi i}\int_\gamma f(z)\,dz$.
- But since $(z-a)^d f(z)$ has the same Laurent series as $f(z)$ except with terms shifted forward by d, we can simply replace $f(z)$ by $(z-a)^d f(z)$ to obtain the $-d-1$th coefficient in the Laurent expansion of $f(z)$.
- In particular, for $d = -1$, we see that the coefficient of $(z-a)^0$, which is to say, the constant term, is given by $\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz$: but this constant term is simply $f(a)$.
- Therefore, we see that for any function $f(z)$ with a convergent Laurent series expansion centered at $z = a$, we have the formula $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz$, where γ is any contour winding once counterclockwise around $z = a$ (for instance, any circle of small positive radius centered at $z = a$).

- Our goal now is to establish this formula, which is known as Cauchy's integral formula, for an arbitrary holomorphic function.

- <u>Theorem</u> (Cauchy's Integral Formula, First Version): Suppose f is holomorphic on a simply connected region and let γ be the counterclockwise boundary of the region. Then for any z_0 in the interior of the region, we have $f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz$.

  - <u>Proof</u>: Since $f(z)$ is holomorphic on the region, $\frac{f(z)}{z-z_0}$ is holomorphic everywhere inside the region except at $z = z_0$.
  - Since z_0 is in the interior of the region, it is some positive distance away from the region's boundary. By our results on deformation of contours, we may deform the contour γ to be a counterclockwise circle γ_r of some sufficiently small radius $r > 0$ centered at z_0 without changing the value of the integral.
  - From our previous calculations, we know that $\int_{\gamma_r} \frac{1}{z-z_0}\,dz = 2\pi i$, so rearranging yields $\frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z_0)}{z-z_0}\,dz = f(z_0)$.
  - Therefore to establish the desired formula, it suffices to show that $\int_{\gamma_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$.
  - Since f is holomorphic, if we define $g(z) = \frac{f(z)-f(z_0)}{z-z_0}$ for $z \neq z_0$ with $g(z_0) = f'(z_0)$, then g is continuous at $z = z_0$ (and elsewhere).
  - Now let M_r be the maximum value of $|f(z) - f(z_0)|$ on γ_r (which exists since f is a continuous function and γ_r is a closed curve), so that for $z \in \gamma_r$ we have $\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \frac{|f(z)-f(z_0)|}{|z-z_0|} \le \frac{M_r}{r}$.
  - Additionally, since f is continuous at $z = z_0$, we have $\lim_{r\to 0} M_r = 0$.
  - Then we have the simple estimate $\left|\int_{\gamma_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \le 2\pi r \cdot \frac{M_r}{r} = 2\pi M_r$, which goes to zero as $r \to 0$.
  - Therefore we have $\lim_{r\to 0+}\int_{\gamma_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$. But by our results on deformation of contours, the value of the integral is the same on each γ_r, so the value must be zero for all γ_r.
  - Thus we have $\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz = \int_{\gamma_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$ and so $\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z_0)}{z-z_0}\,dz = f(z_0)$, as claimed.

- We will remark that by essentially the same proof, we can generalize Cauchy's integral formula to arbitrary closed contours.

- Corollary (Cauchy's Integral Formula, Second Version): Suppose f is holomorphic on a simply connected region R, let γ be any closed contour in R, and let z_0 be any point in the interior of R. Then $\int_\gamma \frac{f(z)}{z-z_0}\,dz = 2\pi i \cdot W_\gamma(z_0) \cdot f(z_0)$.
    - Proof: Using the same estimation argument as in the proof of the first version of Cauchy's integral formula, we see that $\int_\gamma \frac{f(z)}{z-z_0}\,dz = \int_\gamma \frac{f(z_0)}{z-z_0}\,dz$. But the second integral is simply $f(z_0)$ times $\int_\gamma \frac{1}{z-z_0}\,dz = 2\pi i \cdot W_\gamma(z_0)$ by our results on winding numbers.

- Although Cauchy's integral formula may seem to lack great utility at first glance, as we will see, it is really quite a powerful statement.
    - Intuitively, the formula is an averaging result, which we can see very explicitly in the situation where γ is the counterclockwise boundary of the circle of radius r centered at $z = z_0$.
    - Using the parametrization $\gamma(t) = z_0 + re^{it}$ for $0 \le t \le 2\pi$, which has $\gamma'(t) = ire^{it}$, we obtain $\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{it})}{re^{it}}\,ire^{it}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})\,dt$.
    - This last integral is the average value of $f(z)$ on the circle $|z - z_0| = r$, in the usual sense of the average value of a function being its integral on an interval divided the length of the interval.
    - Thus, in short, Cauchy's integral formula says that if f is holomorphic, then the average value of f on any circle is simply equal to the value of f at the center of the circle.
    - More generally, and much more powerfully, it says that the values of a holomorphic function f on the boundary of a simply connected region completely determine the values of f on the interior of the region.
    - This is quite a special property of holomorphic functions, since for general nonholomorphic functions, it is not possible to reconstruct their values on the interior of a region using only their values on the boundary.
    - For example, the nonholomorphic function $f(z) = 1 - z\overline{z} = 1 - |z|^2$ is identically zero on the circle $|z| = 1$ yet is nonzero everywhere inside the circle. So since the function $2f(z)$ has the same values on the circle $|z| = 1$ yet different values everywhere in the interior, it is not possible to reconstruct $f(z)$ for $|z| < 1$ solely from its values on $|z| = 1$, since any such procedure cannot distinguish between $f(z)$ and $2f(z)$.

- The most immediate application of Cauchy's integral formula is to evaluate contour integrals with less effort:

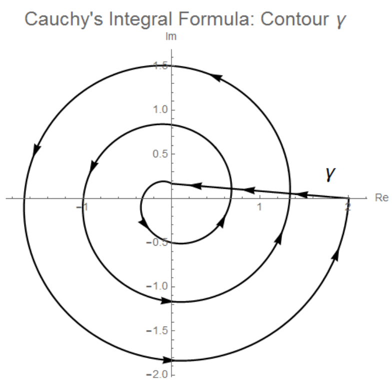
- Example: Find $\int_\gamma \frac{e^z}{z}\,dz$ and $\int_\gamma \frac{e^z}{z-1}\,dz$ where γ is the counterclockwise boundary of the square with vertices $\pm 2 \pm 2i$.
    - Note that each integral is of the form $\int_\gamma \frac{f(z)}{z-z_0}\,dz$ with $f(z) = e^z$ and γ the counterclockwise boundary of a simply connected region.
    - Thus by Cauchy's integral theorem, the first integral equals $2\pi i \cdot f(0) = \boxed{2\pi i}$, while the second integral equals $2\pi i \cdot f(1) = \boxed{2\pi e i}$.
    - Note that we could also evaluate these integrals by deforming the contour γ to an appropriate circle and then using the Laurent series expansions $\frac{e^z}{z} = z^{-1} + 1 + \frac{1}{2!}z + \cdots$ and $\frac{e^z}{z-1} = e(z-1)^{-1} + e + \frac{e}{2!}z + \cdots$, which again yield the two values $2\pi i$ and $2\pi i \cdot e$.

- Example: Find $\int_\gamma \frac{e^z}{z(z-2)}\,dz$ where γ is the counterclockwise boundary of the unit circle.
    - Note that $\frac{e^z}{z(z-2)}$ is holomorphic for all $z \neq 0, 2$, so since $z = 2$ is not inside the unit circle, the integral is of the form $\int_\gamma \frac{f(z)}{z-z_0}\,dz$ with $f(z) = \frac{e^z}{z}$, $z_0 = 0$, and γ the counterclockwise boundary of a simply connected region on which $f(z)$ is holomorphic.

- Thus by Cauchy's integral theorem, the integral equals $2\pi i \cdot f(0) = \boxed{-\pi i}$.

- Example: Find $\int_\gamma \frac{ze^z}{4z+i}\,dz$ where γ is the curve traversing the unit circle twice clockwise.
  - Note that $\frac{ze^z}{4z+i}$ is holomorphic for all $z \neq -i/4$, so if we rewrite the integral as $\int_\gamma \frac{1}{4} \cdot \frac{ze^z}{z+i/4}\,dz$, it is now in the form $\int_\gamma \frac{f(z)}{z-z_0}\,dz$ with $f(z) = ze^z$, $z_0 = -i/4$, and γ a closed contour with winding number -2 around z_0.
  - Thus by Cauchy's integral theorem, the integral equals $2\pi i \cdot f(-i/4) \cdot (-2) = \boxed{-\pi e^{-i/4}}$.

- Example: Find $\int_\gamma \frac{\sin z + e^z}{z(z-1)}\,dz$ where γ is the closed curve plotted below:



  - First, we can use partial fraction decomposition to write $\frac{\sin z + e^z}{z(z-1)} = \frac{\sin z + e^z}{z-1} - \frac{\sin z + e^z}{z}$, so we may equivalently evaluate $\int_\gamma \frac{\sin z + e^z}{z}\,dz$ and $\int_\gamma \frac{\sin z + e^z}{z-1}\,dz$ separately.
  - Each integral is of the form $\int_\gamma \frac{f(z)}{z-z_0}\,dz$ where $f(z) = \sin z + e^z$ is holomorphic everywhere and γ does not pass through z_0.
  - From the plot we can see that γ has winding number 3 around $z = 0$ and winding number 2 around $z = 1$.
  - Therefore, we have $\int_\gamma \frac{\sin z + e^z}{z}\,dz = 2\pi i \cdot W_\gamma(0) \cdot f(0) = 6\pi i$ and $\int_\gamma \frac{\sin z + e^z}{z-1}\,dz = 2\pi i \cdot W_\gamma(1) \cdot f(1) = 4\pi i(\sin 1 + e)$.
  - Hence we see $\int_\gamma \frac{\sin z + e^z}{z(z-1)}\,dz = \int_\gamma \frac{\sin z + e^z}{z-1}\,dz - \int_\gamma \frac{\sin z + e^z}{z}\,dz = \boxed{4\pi i(\sin 1 + e) - 6\pi i}$.

3.2.6 Higher Derivatives and Series Expansions of Holomorphic Functions

- We motivated Cauchy's integral formula by explaining how, if γ winds once counterclockwise around z_0, the integral $\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz$ calculates the constant term in a Laurent series expansion for $f(z)$.
  - By changing the power of $z - z_0$, we can compute other coefficients in the Laurent expansion, and thus obtain values of the various derivatives of $f(z)$ at $z = z_0$.
  - Explicitly, if $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, then $\frac{f(z)}{(z-z_0)^{d+1}} = \sum_{n=-d}^{\infty} a_n (z-z_0)^{n-d-1}$ whose coefficient of $(z-z_0)^{-1}$ is a_d.
  - Therefore, integrating around γ yields $\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{d+1}}\,dz = a_d = \frac{f^{(d)}(z_0)}{d!}$.

◦ We therefore obtain the formula $f^{(d)}(z_0) = \dfrac{d!}{2\pi i}\int_\gamma \dfrac{f(z)}{(z-z_0)^{d+1}}\,dz$, valid whenever $f(z)$ is analytic at $z = z_0$ and for γ a circle of sufficiently small radius to lie inside the disc of convergence for the power series of f at $z = z_0$.

◦ We would now like to prove that this formula holds for any holomorphic function $f(z)$. In fact, we can prove a stronger statement:

- Theorem (Differentiation Via Integration): Suppose R is an open region.

1. Let γ be a closed contour in R and let f be a continuous function on γ. Then for all z_0 not lying on γ, the function $g(z_0) = \int_\gamma \dfrac{f(\zeta)}{\zeta - z_0}\,d\zeta$ is analytic on $R\backslash\gamma$, and furthermore $g^{(n)}(z_0) = n!\int_\gamma \dfrac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta$ for each $n \geq 0$.

   ◦ Proof: Let $z_0 \in R\backslash\gamma$. Then as in the proof of Cauchy's integral formula, since z_0 is in the interior of the region, it is some positive distance r_1 away from the region's boundary, and since z_0 does not lie on the closed contour γ it is some positive distance r_2 away from γ.

   ◦ We show that f has a power series expansion on the disc of radius $r = \min(r_1, r_2)$ centered at z_0.

   ◦ So let $0 < s < r$ and observe the geometric series expansion $\dfrac{1}{\zeta - z} = \dfrac{1}{\zeta - z_0}\cdot\dfrac{1}{1-\dfrac{z-z_0}{\zeta-z_0}} = \sum_{n=0}^\infty \dfrac{1}{(\zeta-z_0)^{n+1}}(z-z_0)^n$.

   ◦ For $\zeta \in \gamma$, this geometric series converges absolutely and uniformly for $|z - z_0| \leq s$ since its common ratio is $\left|\dfrac{z-z_0}{\zeta-z_0}\right| \leq \dfrac{s}{r_2} \leq \dfrac{s}{r} < 1$.

   ◦ Therefore, because $f(\zeta)$ is bounded on γ since it is continuous, we see that the partial sums of $\sum_{n=0}^\infty \dfrac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n$ converges absolutely and uniformly for $|z - z_0| \leq s$ to $\dfrac{g(\zeta)}{\zeta - z}$ for each $\zeta \in \gamma$.

   ◦ Hence by our results on uniform convergence and integrals, we may switch the order of the sum and integral to see

$$\begin{aligned} f(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta &= \int_\gamma \sum_{n=0}^\infty \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n\,d\zeta \\ &= \sum_{n=0}^\infty \int_\gamma \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n\,d\zeta \\ &= \sum_{n=0}^\infty \left[\int_\gamma \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta\right](z-z_0)^n \end{aligned}$$

   which is of the form $g(z) = \sum_{n=0}^\infty a_n(z-z_0)^n$ for $a_n = \int_\gamma \dfrac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta$. This is the desired power series expansion for g and so g is analytic as claimed.

   ◦ The values of the derivatives follow immediately from the formula for the terms a_n and the fact that $g^{(n)}(z_0) = \dfrac{a_n}{n!}$.

2. The function $f(z)$ is holomorphic on R if and only if it is analytic on R.

   ◦ Proof: We have previously shown analytic functions are holomorphic.

   ◦ Conversely, suppose $f(z)$ is holomorphic on R and let $z_0 \in R$.

   ◦ Then by (1), if we take γ to be a counterclockwise circle of radius $r > 0$ inside R centered at z_0, then the function $\int_\gamma \dfrac{f(\zeta)}{\zeta - z}\,d\zeta$ is analytic on $R\backslash\gamma$.

   ◦ But by Cauchy's integral formula, $\int_\gamma \dfrac{f(\zeta)}{\zeta - z}\,d\zeta = 2\pi i\,f(z)$, so in particular f is analytic at z_0 as required.

3. Suppose R is simply connected with counterclockwise boundary γ and f is holomorphic on R. Then f is infinitely differentiable on R and for any z_0 in the interior of R we have $f^{(n)}(z_0) = \dfrac{n!}{2\pi i}\int_\gamma \dfrac{f(z)}{(z-z_0)^{n+1}}\,dz$ for each $n \geq 0$.
   - Proof: By (2), if f is holomorphic then f is analytic, and analytic functions are infinitely differentiable as we have previously shown.
   - For the formulas observe as in (2) that $f = \dfrac{1}{2\pi i} g$ via the Cauchy integral formula, where g is the function from (1), and use the formulas for the derivatives in (1).
4. Suppose that f is holomorphic on the closed disc $|z - z_0| \leq r$. Then the power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for f centered at $z = z_0$ has radius of convergence at least r.
   - Proof: By (2) we know that f is analytic at z_0 and (3) gives $a_n = \dfrac{1}{2\pi i}\int_\gamma \dfrac{f(z)}{(z-z_0)^{n+1}}\,dz$ where we may take γ to be the counterclockwise circle $|z - z_0| = r$.
   - Since f is continuous it is bounded on γ, say with $|f(z)| \leq M$.
   - Then for $|z - z_0| = r$ we have $\left|\dfrac{f(z)}{(z-z_0)^{n+1}}\right| = \dfrac{|f(z)|}{r^{n+1}} \leq \dfrac{M}{r^{n+1}}$, so since γ has length $2\pi r$ we obtain the estimate $|a_n| = \left|\dfrac{1}{2\pi i}\int_\gamma \dfrac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \leq \dfrac{1}{2\pi} \cdot 2\pi r \cdot \dfrac{M}{r^{n+1}} = \dfrac{M}{r^n}$.
   - Thus we see $\limsup_{n\to\infty} |a_n|^{1/n} \leq 1/r$ and so the radius of convergence of the power series is at least r, as claimed.
5. (Morera's Theorem) If f is continuous on the region R and $\int_\gamma f(z)\,dz = 0$ for all closed contours γ in R, then f is holomorphic on R.
   - Recall that we previously established the converse of this theorem (if f is holomorphic then its integral is zero on all closed contours).
   - Proof: By our earlier result on existence of antiderivatives, since f is continuous and $\int_\gamma f(z)\,dz = 0$ on all closed contours, f possesses an antiderivative F on R, with F holomorphic and $F'(z) = f(z)$.
   - But then by (3) since F is holomorphic it is infinitely differentiable. In particular, F is twice differentiable, meaning that $F' = f$ is differentiable, which is to say that f is holomorphic.

- In addition to the rather unexpected fact that we can compute the derivative of a holomorphic function via integration, the fact that every holomorphic function is infinitely differentiable is also quite unexpected.
  - This state of affairs stands in stark contrast to the situation with real-valued functions, since for example the function $f(x) = x^{n+2/3}$ is differentiable n times at $x = 0$ but not $n+1$ times: the $(n+1)$st derivative tends to $-\infty$ as $x \to 0-$ and to $+\infty$ as $x \to 0+$)
- In the next chapter we will discuss many additional applications of Cauchy's integral formula that expand on the results we have obtained so far.

Well, you're at the end of my handout. Hope it was helpful.