Complex Analysis (part 2): Complex Power Series (by Evan Dummit, 2022, v. 1.00)

Contents

2 Complex Power Series

In this chapter, our goal is to study power series in the context of complex-valued functions, with the primary motivation being to give a new general method for constructing holomorphic functions. Indeed, as we will see in the next chapter, in fact every holomorphic function on a region R can be written locally as a power series, and so this method does in some sense encompass every possible holomorphic function.

We first develop the general theory of formal power series and formal Laurent series with complex coefficients, which provide tools for manipulating and computing power series expansions. We next study questions of power series convergence, continuity, and differentiability to show that any power series with a positive radius of convergence is always holomorphic. Finally, we then apply these results to define various elementary functions such as the complex exponential e^z and the trigonometric functions $\sin(z)$ and $\cos(z)$ as convergent power series in order to show that these functions' familiar properties carry over quite naturally from $\mathbb{R}$ to $\mathbb{C}$.

2.1 Formal Power Series

- In single-variable calculus, we have various familiar expansions of functions as convergent power series, such as $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ and $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$, where x represents some indeterminate real number.
  - We may perform various algebraic operations on such series, such as adding and subtracting them, multiplying them, and in some cases even dividing them, to obtain new series. For example, we may compute formally using the distributive law that

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \left[1+x+x^2+x^3+\cdots\right]\left[1+x+x^2+x^3+\cdots\right] \\
&= \left[1+x+x^2+x^3+\cdots\right] + x\left[1+x+x^2+x^3+\cdots\right] + x^2\left[1+x+x^2+x^3+\cdots\right] + x^3\left[1+x+x^2+x^3+\cdots\right] + \cdots \\
&= \left[1+x+x^2+x^3+\cdots\right] + \left[x+x^2+x^3+x^4+\cdots\right] + \left[x^2+x^3+x^4+x^5+\cdots\right] + \left[x^3+x^4+x^5+x^6+\cdots\right] + \cdots \\
&= 1+2x+3x^2+4x^3+5x^4+\cdots = \sum_{n=0}^{\infty}(n+1)x^n.
\end{aligned}$$

- The main observation is that the calculation above is purely formal, in the sense that it entirely ignores any questions about convergence of the various power series involved and requires only manipulation of expressions involving the indeterminate x.
- Our goal now is to formulate rigorously this notion of "formal power series" along with the various algebraic operations we may perform on them. For no additional cost, we will also work with complex coefficients rather than purely real ones.

2.1.1 Formal Power Series With Complex Coefficients

- Definition: Let z be an indeterminate. A formal power series with complex coefficients in z is an expression of the form $\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots$ for any choice of coefficients $a_i \in \mathbb{C}$. The variable z represents an indeterminate, and does not refer to a complex number. Two formal power series are equal precisely when all of their coefficients a_i are equal.
  - The term "indeterminate" is deliberately undefined in the definition above. A more concrete (but vastly less intuitive) definition of formal power series can be given using Cartesian products[1], but we will not use it.
  - As usual, we will omit writing terms with zero coefficients, and omit coefficients of 1, except for emphasis, just as we do for polynomials and regular series. Thus, we would abbreviate the series $0 + 1z + 1z^2 + 1z^3 + \cdots = \sum_{n=1}^{\infty} z^n$ as $z + z^2 + z^3 + \cdots$.
  - Examples: Some examples of formal power series are $\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$, $\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + z^2 + z^3 + \cdots$, and $\sum_{n=1}^{\infty} n^n z^n = z + 4z^2 + 27z^3 + 256z^4 + \cdots$.
  - Example: A formal power series with all but finitely many coefficients equal to zero is simply a polynomial in z (and vice versa). Thus, $1+z$, $1+9z-z^{2022}$, and $1+z+z^2+\cdots+z^{999999}$ are all formal power series.
  - Example: Any complex number c is trivially a formal power series: $c = c + 0z + 0z^2 + \cdots$.
- We may manipulate formal power series by adding and multiplying them in the same way that we add and multiply abstract polynomials (indeed, the idea is that formal power series behave essentially like "polynomials that continue forever").
  - Addition is defined termwise: $(a_0 + a_1 z + a_2 z^2 + \cdots) + (b_0 + b_1 z + b_2 z^2 + \cdots) = (a_0 + b_0) + (a_1 + b_1)z + (a_2 + b_2)z^2 + \cdots$, or, formally, as $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (a_n + b_n) z^n$.
  - Likewise, subtraction and scaling by a complex constant are also defined termwise: we set $\sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (a_n - b_n) z^n$ and for any $c \in \mathbb{C}$ we set $c \cdot \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (c a_n) z^n$.
  - Multiplication is defined first on "monomials" (power series with only one nonzero coefficient), via $(a_n z^n) \cdot (b_m z^m) = a_n b_m z^{n+m}$, and then extended to arbitrary power series via the distributive laws. Explicitly, one obtains

$$(a_0 + a_1 z + a_2 z^2 + \cdots) \cdot (b_0 + b_1 z + b_2 z^2 + \cdots) = a_0 b_0 + (a_1 b_0 + a_0 b_1) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 + \cdots$$

    where the coefficient of z^n in the product is given by $\sum_{k=0}^{n} a_k b_{n-k}$. More formally, we have $[\sum_{n=0}^{\infty} a_n z^n] \cdot [\sum_{n=0}^{\infty} b_n z^n] = \sum_{n=0}^{\infty} [\sum_{k=0}^{n} a_k b_{n-k}] z^n$.
  - We note that all of the coefficients in any product of formal power series are given by finite sums, and thus we do not have any questions of convergence arising here.
- We also have a notion of "order" for formal power series, which has similar properties to the degree of a polynomial:
- Definition: The order of a formal power series $\sum_{n=0}^{\infty} a_n z^n$ is the smallest integer n for which $a_n \neq 0$. (By convention, the order of the zero power series is ∞.)

[1] Specifically: inside the Cartesian product of copies of identical copies of $\mathbb{C}$ indexed by the nonnegative integers, namely $\prod_{\mathbb{Z}_{\geq 0}} \mathbb{C} = \{(a_0, a_1, a_2, \dots)\}_{a_i \in \mathbb{C}}$, we define the formal power series $\sum_{n=0}^{\infty} a_n z^n$ to be the sequence $(a_0, a_1, a_2, \dots, a_n, \dots)$. The indeterminate z then corresponds to the sequence $(0, 1, 0, 0, 0, \dots, 0, \dots)$, which is not a complex number but rather an element of this Cartesian product!

- Examples: The formal power series $z^4 + 4z^5 + 27z^6 + 256z^7 + \cdots$ has order 4, while the series $z^3 + z^9$ has order 3, and the series $1 + z + z^2 + z^3 + \cdots$ has order 0.
- It is not hard to verify from the definitions of addition and multiplication that for any formal power series f and g, we have $\mathrm{ord}(fg) = \mathrm{ord}(f) + \mathrm{ord}(g)$ and $\mathrm{ord}(f+g) \geq \min(\mathrm{ord}(f), \mathrm{ord}(g))$.

- Example: For the formal power series $f = \sum_{n=0}^{\infty}(n+1)z^n = 1+2z+3z^2+4z^3+5z^4+\cdots$ and $g = \sum_{n=0}^{\infty} n^2 z^n = z + 4z^2 + 9z^3 + 16z^4 + \cdots$, find the terms up to order 4 for $4f$, $f+g$, $f-g$, fg, and f^2.
  - Scaling all terms by 4 yields $4f = \sum_{n=0}^{\infty} 4(n+1)z^n = \boxed{4 + 8z + 12z^2 + 16z^3 + 20z^4 + \cdots}$.
  - Adding corresponding terms yields $f + g = \sum_{n=0}^{\infty}(n+1+n^2)z^n = \boxed{1 + 3z + 7z^2 + 12z^3 + 21z^4 + \cdots}$.
  - Subtracting corresponding terms yields $f - g = \sum_{n=0}^{\infty}(n+1-n^2)z^n = \boxed{1 + z - z^2 - 5z^3 - 11z^4 - \cdots}$.
  - For the product fg, one may compute that the order-n coefficient is $\sum_{k=0}^{n}(k+1)(n-k)^2 = \frac{1}{12}n(n+1)^2(n+2)$ (the formula for this sum can be verified by induction). But it is easier to simply multiply out the terms explicitly and then discard terms of order larger than 4:

$$\begin{aligned}
fg &= f(z + 4z^2 + 9z^3 + 16z^4 + \cdots) \\
&= f \cdot z + f \cdot 4z^2 + f \cdot 9z^3 + f \cdot 16z^4 + \cdots \\
&= (z + 2z^2 + 3z^3 + 4z^4 + \cdots) + (4z^2 + 8z^3 + 12z^4 + \cdots) + (9z^3 + 18z^4 + \cdots) + (16z^4 + \cdots) + \cdots \\
&= \boxed{z + 6z^2 + 20z^3 + 50z^4 + \cdots}.
\end{aligned}$$

  - Likewise, for f^2 we can find an explicit formula for the order-n coefficient as $\sum_{k=0}^{n}(k+1)(n-k+1) = \frac{1}{6}(n+1)(n+2)(n+3)$, or simply multiply out the expansions:

$$\begin{aligned}
f^2 &= f(1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots) \\
&= f \cdot 1 + f \cdot 2z + f \cdot 3z^2 + f \cdot 4z^3 + f \cdot 5z^4 + \cdots \\
&= (1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots) + (2z + 4z^2 + 6z^3 + 8z^4 + \cdots) + (3z^2 + 6z^3 + 9z^4 + \cdots) + (4z^3 + 8z^4 + \cdots \\
&= \boxed{1 + 4z + 10z^2 + 20z^3 + 35z^4 + \cdots}.
\end{aligned}$$

- These operations on formal power series obey the usual properties of arithmetic:

- Proposition (Formal Power Series Arithmetic): Suppose that a, b, and c are formal power series with complex coefficients in z. Then the following hold:

  1. Addition is associative: $a + (b + c) = (a + b) + c$.
  2. Addition is commutative: $a + b = b + a$.
  3. The power series 0 is an additive identity: $a + 0 = a$.
  4. If $a = \sum_{n=0}^{\infty} a_n z^n$ then the power series $-a = \sum_{n=0}^{\infty}(-a_n)z^n$ is an additive inverse of a: $a + (-a) = 0$.
  5. Multiplication is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
  6. Multiplication is commutative: $a \cdot b = b \cdot a$.
  7. The power series 1 is a multiplicative identity: $1 \cdot a = a$.
  8. Multiplication distributes over addition: $a \cdot (b + c) = a \cdot b + a \cdot c$.
     - Proofs: These all follow immediately from the definitions of power series addition and multiplication along with some tedious algebra and the corresponding properties (associativity of addition, etc.) of the complex numbers.
     - Remark (for those who like ring theory): This proposition shows that the set of complex formal power series is a commutative ring with 1. This ring is denoted $\mathbb{C}[[z]]$ (the double brackets indicate power series, to emphasize the similarity with the polynomial ring $\mathbb{C}[z]$).

- Notice that the proposition above establishes that the formal power series possess all of the properties of being a field except for the existence of multiplicative inverses. Indeed, some formal power series have a multiplicative inverse while others do not.
  - Example: We have $(1-z)(1+z+z^2+z^3+\cdots) = \left[1+z+z^2+z^3+\cdots\right]+(-z)\left[1+z+z^2+z^3+\cdots\right] = \left[1+z+z^2+z^3+\cdots\right]+\left[-z-z^2-z^3-z^4-\cdots\right] = 1$, and so the power series $1-z$ and $1+z+z^2+z^3+\cdots$ are multiplicative inverses of one another.
  - Example: We claim that the formal power series $z+z^2$ has no multiplicative inverse: if it did, say $b_0+b_1z+b_2z^2+\cdots$, multiplying out would yield $1=(z+z^2)(b_0+b_1z+b_2z^2+\cdots) = b_0z+(b_0+b_1)z^2+(b_1+b_2)z^3+\cdots$, but this is impossible since the constant term of the product is not equal to 1.
  - The issue in the second example is that the power series $z+z^2$ has a zero constant term, and thus all terms in any product involving this power series have order at least 1 in z, hence in particular will also have a constant term of zero. In fact, this is the only obstruction to having a multiplicative inverse:

- Proposition (Multiplicative Inverses of Power Series): The complex power series $\sum_{n=0}^{\infty} a_n z^n$ has a multiplicative inverse if and only if its order is 0 (equivalently, when its constant term $a_0 \neq 0$), and in such a case its inverse is unique and given by $\sum_{n=0}^{\infty} b_n z^n$ where $b_0 = a_0^{-1}$ and $b_n = -a_0^{-1}\sum_{k=1}^{n} a_k b_{n-k}$ for all $n \geq 1$.
  - We will remark that the recurrence relation given for the b_i is often rather cumbersome to solve in practice. The primary utility of this result is the characterization of the power series with multiplicative inverses.
  - Proof: Suppose that $\sum_{n=0}^{\infty} a_n z^n$ has a multiplicative inverse $\sum_{n=0}^{\infty} b_n z_n$, so that the product $[\sum_{n=0}^{\infty} a_n z^n]\cdot[\sum_{n=0}^{\infty} b_n z^n]$ equals 1.
  - Since the product is $\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} a_k b_{n-k}\right] z^n$, comparing coefficients yields $a_0b_0 = 1$, $a_0b_1+a_1b_0 = 0$, $a_0b_2+a_1b_1+a_2b_0 = 0$, ... , and in general $\sum_{k=0}^{n} a_k b_{n-k} = 0$ for all $n \geq 1$.
  - In particular, we cannot have $a_0 = 0$ since this would contradict the first equation. On the other hand, if $a_0 \neq 0$, then the first equation uniquely determines $b_0 = a_0^{-1}$, the second equation uniquely determines $b_1 = -a_0^{-1}[a_1b_0]$, the third equation uniquely determines $b_2 = -a_0^{-1}[a_1b_1+a_2b_0]$, ... , and the nth equation uniquely determines $b_n = -a_0^{-1}\sum_{k=1}^{n} a_k b_{n-k}$.
  - Therefore, we conclude that $\sum_{n=0}^{\infty} a_n z^n$ has a multiplicative inverse if and only if $a_0 \neq 0$ and that its inverse is as claimed.

- Example: Find the terms up to order 4 in the multiplicative inverse of the formal power series $f = \sum_{n=0}^{\infty}(n+1)z^n = 1+2z+3z^2+4z^3+5z^4+\cdots$.
  - Per the proposition, both f and g do have multiplicative inverses since their constant terms are nonzero.
  - Suppose the inverse of f is $b_0+b_1z+b_2z^2+b_3z^3+b_4z^4+\cdots$. Then we have $(1+2z+3z^2+4z^3+5z^4+\cdots)(b_0+b_1z+b_2z^2+b_3z^3+b_4z^4+\cdots) = 1$ which upon expanding yields
  $$b_0+(2b_0+b_1)z+(3b_0+2b_1+b_2)z^2+(4b_0+3b_1+2b_2+b_3)z^3+(5b_0+4b_1+3b_2+2b_3+b_4)z^4+\cdots = 1.$$
  - Comparing coefficients gives $b_0 = 1$, $2b_0+b_1 = 0$ so that $b_1 = -2$, $3b_0+2b_1+b_2 = 0$ so that $b_2 = 1$, $4b_0+3b_1+2b_2+b_3 = 0$ so that $b_3 = 0$, and $5b_0+4b_1+3b_2+2b_3+b_4 = 0$ so that $b_4 = 0$.
  - Thus we see $f^{-1} = \boxed{1-2z+z^2+0z^3+0z^4+\cdots}$.
  - Remark: In fact, as one may check, all of the subsequent terms in f^{-1} are zero, meaning that $f^{-1} = 1-2z+z^2$ exactly.

- We can also give another procedure to find the multiplicative inverse of a power series $f = \sum_{n=0}^{\infty} a_n z^n$ with $a_0 \neq 0$.
  - First, since $f^{-1} = a_0^{-1}(f/a_0)^{-1}$, we may reduce to calculating the inverse of f/a_0, which is easier since the constant term is now 1.
  - So write $g = f/a_0$ where $g = 1+\sum_{n=1}^{\infty} b_n z^n$. If we write $b = -\sum_{n=1}^{\infty} b_n z^n$, then we need only compute the formal power series inverse of $1-b$.

- But in our calculation earlier we saw that the formal power series inverse of $1-b$ was $1+b+b^2+b^3+\cdots$, and this latter sum is well defined because b has constant term 0 (so only the terms up to b^n will contribute to the sum representing the coefficient of z^n).
- Therefore, expanding out this sum yields the inverse of $1-b$, and then scaling by a_0^{-1} yields the inverse of $\sum_{n=0}^{\infty} a_n z^n$.

- Example: Find the terms up to order 4 in the multiplicative inverse of the formal power series $f = 2-4z-4z^2$.
  - Here we have $a_0 = 2$ so dividing by 2 yields $g = 1-2z-2z^2$ and then $b = 2z+2z^2$.
  - Then the formal power series inverse of $g = 1-b$ is

$$\begin{aligned}1+b+b^2+b^3+b^4+\cdots &= 1+(2z+2z^2)+(2z+2z^2)^2+(2z+2z^2)^3+(2z+2z^2)^4+\cdots \\ &= 1+(2z+2z^2)+(4z^2+8z^3+4z^4)+(8z^3+24z^4+\cdots)+(16z^4+\cdots)+\cdots \\ &= 1+2z+6z^2+16z^3+44z^4+\cdots\end{aligned}$$

which upon rescaling yields $f^{-1} = \boxed{\frac{1}{2}+z+3z^2+8z^3+22z^4+\cdots}$.

  - To check we can multiply out $f \cdot f^{-1} = (2-4z-4z^2)(\frac{1}{2}+z+3z^2+8z^3+22z^4) = 1-120z^5-88z^6$, which indeed is 1 up through degree 4.

2.1.2 Formal Laurent Series With Complex Coefficients

- We have seen so far that the formal power series give a natural extension of polynomials, in that they possess most of the familiar algebraic properties of polynomials, but also allow us to compute multiplicative inverses in many cases.
  - Indeed, we may even rewrite many rational functions of z as power series by calculating multiplicative inverses.
  - For example, since $(1-z)^{-1} = 1+z+z^2+z^3+\cdots$, we may by extension write $\frac{1+z}{1-z} = (1+z)(1-z)^{-1} = (1+z)(1+z+z^2+z^3+\cdots) = 1+2z+2z^2+2z^3+\cdots$.
  - However, not all rational functions of z may be written as a power series in this way, since we require the denominator series to have a multiplicative inverse.
  - For example, we cannot expand $\frac{1+3z}{z-z^2}$ as a formal power series since the denominator $z-z^2$ is not invertible, nor can we even expand the very simple rational function $\frac{1}{z}$.
- Our ultimate goal is to use power series as a tool for studying differentiability by shifting focus from formal series to convergent series, where we allow "plugging in" values $z \in \mathbb{C}$ to our series to obtain a complex-valued function.
  - Because rational functions are differentiable, we would like to be able to study all of them from the viewpoint of series together, rather than having to make consistent exceptions for series whose denominator is not invertible.
- Pleasantly, our earlier analysis of non-invertible series already provides an avenue for handling this difficulty: we simply need to allow powers of z to have multiplicative inverses as well.
  - In other words, we must also allow the series of the form z^{-n} for each fixed positive integer n.
  - Since we wish to retain all of the algebraic properties of series (i.e., we want to be able to add and multiply them), this requirement is equivalent to allowing our series to have a finite number of terms with negative exponents of z.
  - Such series are called formal Laurent series:

- Definition: Let z be an indeterminate. A formal Laurent series with complex coefficients in z is an expression of the form $\sum_{n=-d}^{\infty} a_n z^n = a_{-d}z^{-d} + a_{1-d}z^{1-d} + \cdots + a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$ for any choice of coefficients $a_i \in \mathbb{C}$ and some fixed nonnegative integer d. Two formal Laurent series are equal precisely when all of their coefficients a_i are equal (with the convention that any terms not included have an implicit coefficient of 0).
  - Every formal power series is automatically a formal Laurent series with $d = 0$.
  - We use the same definition of order for formal Laurent series: explicitly, the order of the Laurent series $\sum_{n=-d}^{\infty} a_n z^n$ is the smallest integer n for which $a_n \neq 0$ (where as before, the order of the zero series is ∞).
  - Examples: Some other examples of formal Laurent series are z^{-1} of order -1, $\sum_{n=-4}^{\infty} z^n = z^{-4} + z^{-3} + z^{-2} + z^{-1} + 1 + z + \cdots$ of order -4, $\sum_{n=-1}^{\infty} nz^n = -z^{-1} + z + 2z^2 + 3z^3 + \cdots$ of order -1, $\sum_{n=-3}^{\infty} \dfrac{z^n}{|n|!} = \dfrac{z^{-3}}{3!} + \dfrac{z^{-2}}{2!} + z^{-1} + 1 + z + \dfrac{z^2}{2!} + \cdots$ of order -3, and $7z^{-2022} + (1-i)z^{-3} + 2iz^4 + z^{88888}$ of order -2022.

- Intuitively, we can think of a formal Laurent series as merely being a formal power series times some power of z^{-1}. More explicitly, we have $\sum_{n=-d}^{\infty} a_n z^n = z^{-d} \cdot [\sum_{n=0}^{\infty} a_n z^{n+d}] = z^{-d} \cdot [a_{-d} + a_{1-d}z + \cdots + a_0 z^d + a_1 z^{d+1} + \cdots]$.
  - From this standpoint, we see that we can manipulate formal Laurent series by adding and multiplying them in precisely the same way that we do with formal power series.
  - Addition, subtraction, and scaling are defined termwise: $\sum_{n=-d}^{\infty} a_n z^n \pm c \cdot \sum_{n=-d}^{\infty} b_n z^n = \sum_{n=-d}^{\infty} (a_n \pm cb_n) z^n$. (If the leading negative-power terms do not have the same degree we simply extend one of the series with coefficients of 0 until it matches the other.)
  - Multiplication is defined via the distributive law just as before: first we set $(a_n z^n) \cdot (b_m z^m) = a_n b_m z^{n+m}$ for any integers m and n (now possibly negative), and then we extend to arbitrary Laurent series via the distributive laws. Explicitly, one obtains

$$(a_{-d}z^{-d} + \cdots + a_0 + a_1 z + \cdots) \cdot (b_{-e}z^{-e} + \cdots + b_0 + b_1 z + \cdots) = a_{-d}b_{-e}z^{-d-e} + (a_{-d}b_{1-e} + a_{1-d}b_{-e})z^{1-d-e} + \cdots$$

    where the coefficient of z^n in the product $\left[\sum_{n=-d}^{\infty} a_n z^n\right] \cdot \left[\sum_{n=-e}^{\infty} b_n z^n\right]$ is given by $\sum_{k=-d}^{n+e} a_k b_{n-k}$.
  - Alternatively, we could factor out the negative powers of z first and then multiply the remaining terms as regular formal power series: $\left[\sum_{n=-d}^{\infty} a_n z^n\right] \cdot \left[\sum_{n=-e}^{\infty} b_n z^n\right] = z^{-d}[\sum_{n=0}^{\infty} a_n z^{n+d}] \cdot z^{-e}[\sum_{n=0}^{\infty} b_n z^{n+e}] = z^{-d-e} \cdot [\sum_{n=0}^{\infty} a_n z^{n+d}][\sum_{n=0}^{\infty} b_n z^{n+e}]$.
  - Because we only have a finite number of negative-exponent terms, all of the the coefficients still only require finite sums, so we do not need to worry about convergence issues.

- Formal Laurent series have all of the same algebraic properties as formal power series, but with one substantial added bonus: every nonzero Laurent series has a multiplicative inverse.

- Proposition (Formal Power Series Arithmetic): The formal Laurent series with complex coefficients form a field under addition and multiplication. More explicitly, suppose that a, b, and c are formal Laurent series with complex coefficients in z. Then the following hold:
  1. Addition is associative: $a + (b + c) = (a + b) + c$.
  2. Addition is commutative: $a + b = b + a$.
  3. The Laurent series 0 is an additive identity: $a + 0 = a$.
  4. If $a = \sum_{n=-d}^{\infty} a_n z^n$ then the Laurent series $-a = \sum_{n=-d}^{\infty} (-a_n) z^n$ is an additive inverse of a: $a + (-a) = 0$.
  5. Multiplication is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
  6. Multiplication is commutative: $a \cdot b = b \cdot a$.
  7. The Laurent series 1 is a multiplicative identity: $1 \cdot a = a$.
  8. Multiplication distributes over addition: $a \cdot (b + c) = a \cdot b + a \cdot c$.
  9. Every nonzero Laurent series a has a multiplicative inverse a^{-1} with $a^{-1} \cdot a = 1$, and $\mathrm{ord}(a^{-1}) = -\mathrm{ord}(a)$.

- Properties (1)-(8) all follow in the same way as for formal power series. The main point of interest is (9), which follows from our work earlier on invertible power series. Intuitively, the idea is just to factor out an appropriate power of z and then observe that the "leftover" formal power series with nonzero constant term is invertible.
- Proof (9): Suppose $a = \sum_{n=d}^{\infty} a_n z^n$ is a nonzero Laurent series of order d (where d may be positive, negative, or zero).
- Then $a = z^d \cdot \sum_{n=0}^{\infty} a_{n+d} z^n$. But the formal power series $\tilde{a} = \sum_{n=0}^{\infty} a_{n+d} z^n = a_d + a_{1+d} z + \cdots$ has nonzero constant term, so it has some multiplicative inverse power series b with $\tilde{a} \cdot b = 1$.
- Then $a \cdot (z^{-d} b) = (z^d \tilde{a}) \cdot z^{-d} b = \tilde{a} \cdot b = 1$, which means the series $z^{-d} b$ is a formal Laurent series inverse for a as required.
- The statement about the order follows immediately, since $\text{ord}(a^{-1}) = \text{ord}(z^{-d} b) = -d = -\text{ord}(a)$.
- Remark (for those who like ring theory): The field of formal Laurent series is denoted $\mathbb{C}((z))$ to highlight the analogy with the field of rational functions $\mathbb{C}(z)$: indeed, $\mathbb{C}((z))$ is the field of fractions of the formal power series ring $\mathbb{C}[[z]]$.

- Example: Find the terms up to order 5 in the multiplicative inverse of the formal Laurent series $f = z + z^3$.
  - By the discussion above, we simply extract the appropriate power of z from f and then invert the remaining portion.
  - We factor $f = z(1 + z^2)$ and then must find the formal power series inverse of $1 + z^2$, which is $\sum_{n=0}^{\infty} (-z^2)^n = 1 - z^2 + z^4 - z^6 + z^8 - \cdots$.
  - Then $f^{-1} = z^{-1}(1 + z^2)^{-1} = z^{-1} \sum_{n=0}^{\infty} (-z^2)^n = \boxed{z^{-1} - z + z^3 - z^5 + \cdots}$.

- Example: Find the multiplicative inverse of the formal Laurent series $f = \sum_{n=-2}^{\infty} z^n = z^{-2} + z^{-1} + 1 + z + z^2 + \cdots$.
  - As above, we simply extract the appropriate power of z from f and then invert the remaining portion.
  - We factor $f = z^{-2} \sum_{n=0}^{\infty} z^n$ and then must find the formal power series inverse of $\sum_{n=0}^{\infty} z^n$, which as we have seen previously is $1 - z$.
  - Then $f^{-1} = z^2 (\sum_{n=0}^{\infty} z^n)^{-1} = z^2(1 - z) = \boxed{z^2 - z^3}$.

- Example: Find the terms up to order 5 in the multiplicative inverse of the formal Laurent series $f = \sum_{n=-1}^{\infty} n^2 z^n = z^{-1} + z + 4z^2 + 9z^3 + \cdots$.
  - We factor $f = z^{-1}(1 + z^2 + 4z^3 + 9z^4 + \cdots)$ and then must find the formal power series inverse of $1 + z^2 + 4z^3 + 9z^4 + \cdots$, which we do term-by-term.
  - Since we want $f^{-1} = z(1 + z^2 + 4z^3 + 9z^4 + \cdots)^{-1}$ up to order 5, we only need to calculate $(1 + z^2 + 4z^3 + 9z^4 + \cdots)^{-1}$ up to order 4.
  - With $b = -z^2 - 4z^3 - 9z^4 - \cdots$ we see that the inverse of $1 - b$ is

$$\begin{aligned} 1 + b + b^2 + b^3 + b^4 + \cdots &= 1 + (-z^2 - 4z^3 - 9z^4 - \cdots) + (-z^2 - 4z^3 - 9z^4 - \cdots)^2 + (-z^2 - 4z^3 - 9z^4)^3 + \cdots \\ &= (-z^2 - 4z^3 - 9z^4 - \cdots) + (z^4 + 8z^5 + \cdots) + (-z^6 + \cdots) + \cdots \\ &= 1 - z^2 - 4z^3 - 8z^4 + \cdots. \end{aligned}$$

  - Then up to order 5, we have $f^{-1} = z(1 - z^2 - 4z^3 - 8z^4 + \cdots) = \boxed{z - z^3 - 4z^4 - 8z^5 + \cdots}$.

- By computing the Laurent expansion of the inverse of the denominator, we can express any rational function in z as a formal Laurent series (indeed, we can even express the quotient of any two Laurent series as another Laurent series as long as the denominator is not zero).

- Example: Expand $\dfrac{1 - 3z}{z + z^3}$ as a formal Laurent series up to order 5.

* Using the inverse of $z+z^3$ calculated above, we have

$$\begin{aligned}\frac{1-3z}{z+z^3} &= (1-3z)(z+z^3)^{-1} = (1-3z)(z^{-1}-z+z^3-z^5+\cdots) \\ &= (z^{-1}-z+z^3-z^5+\cdots)+(-3+3z^2-3z^4+3z^6-\cdots) \\ &= \boxed{z^{-1}-3-z+3z^2+z^3-3z^4-z^5+\cdots}.\end{aligned}$$

2.2 Convergence of Complex Power Series

- So far, we have discussed power series and Laurent series in z as purely formal objects. Our goal now is to study power series as functions of the complex variable z, which amounts to "plugging in" a complex number in for z.
  - The main issue at hand is that for an arbitrary power series or Laurent series, the resulting series expression may not converge for all (or even any) values $z \in \mathbb{C}$.
  - Therefore, we will first discuss convergence of sequences and series of complex numbers, and then we will apply these results to analyze the convergence properties of power series and Laurent series.

2.2.1 Sequences and Series of Complex Numbers

- We begin with the definition of a convergent sequence of complex numbers. The definition is essentially the same as for sequences of real numbers:
- Definition: We say a sequence $\{a_n\}_{n\geq 1}$ of complex numbers converges to the limit L, written $\lim_{n\to\infty} a_n = L$, if for any $\epsilon > 0$ there exists a positive integer N such that for all $n \geq N$ it is true that $|a_n - L| < \epsilon$.
  - A sequence converges (with no qualifier) if it converges to some limit L, and it fails to converge if there exists no such L.
  - As usual, the intuition is that the terms a_n approach L "arbitrarily closely" as n grows large.
  - The above definition simply makes this notion precise: namely, for any "error amount" $\epsilon > 0$, we can always specify how far out N in the sequence we must go to ensure that all of the terms afterwards (i.e., with $n \geq N$) are within the error amount of the limit L.
  - Example: The sequence with $a_n = 1/n$ for all $n \geq 1$ converges to 0 as $n \to \infty$, since for any $\epsilon > 0$, if we take any integer $N > 1/\epsilon$ then for $n \geq N$ we have $|a_n - L| = |1/n - 0| \leq 1/N < \epsilon$ as required.
  - Example: The sequence with $a_n = i^n$ for all $n \geq 1$ does not converge, since the terms cycle among i, -1, $-i$, 1 and so for $\epsilon = 1/2$ there are no possible values of L and N with $|a_n - L| < 1/2$ for all $n \geq N$, since by the triangle inequality we would have $\sqrt{2} = |a_n - a_{n+1}| \leq |a_n - L| + |L - a_{n+1}| < 1$ and this is impossible.
- There are many basic properties of convergence of sequences, some of which we record here:
- Proposition (Properties of Sequence Convergence): Suppose $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are complex sequences.
  1. If $\lim_{n\to\infty} a_n = L_a$ and $\lim_{n\to\infty} b_n = L_b$, then $\lim_{n\to\infty}(a_n + b_n) = L_a + L_b$, $\lim_{n\to\infty}(a_n - b_n) = L_a - L_b$, $\lim_{n\to\infty}(a_n b_n) = L_a L_b$, and finally if $L_b \neq 0$ then $\lim_{n\to\infty}(a_n/b_n) = L_a/L_b$.
  2. If the sequence $\{a_n\}_{n\geq 1}$ converges to L, then any subsequence (i.e., a sequence $\{a_{f(n)}\}_{n\geq 1}$ for some strictly increasing integer-valued function $f(n)$) also converges to L.
  3. If $f : \mathbb{R} \to \mathbb{C}$ is any function and $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n)$ exists and equals L.
  4. We have $\lim_{n\to\infty} a_n = x + iy$ if and only if $\lim_{n\to\infty} \mathrm{Re}(a_n) = x$ and $\lim_{n\to\infty} \mathrm{Im}(a_n) = y$.
  5. The function $f : \mathbb{C} \to \mathbb{C}$ is continuous if and only if $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$ for every convergent sequence $\{a_n\}_{n\geq 1}$.
  6. The sequence $\{a_n\}_{n\geq 1}$ converges if and only if for any $\epsilon > 0$ there exists an N such that for all $m, n \geq N$ it is true that $|a_m - a_n| < \epsilon$. (A sequence with this latter property is called a Cauchy sequence.)

- All of these results are natural analogues of the corresponding properties of real-valued sequences, and are fairly direct applications of the definition of sequence convergence.
- We omit the proofs, but make some remarks: (1) establishes that limits of sequences have all of the usual algebraic properties shared by limits of functions, and (3) establishes that we can compute limits of sequences using properties of limits of functions.
- Also, (4) establishes that we can reduce any limit about complex sequences to a pair of real-valued limits, while (5) establishes the so-called sequential definition of continuity (it is essentially another form of the substitution rules for limits of functions).
- Finally, for (6): intuitively, a convergent sequence is one where the terms approach some fixed number L, while a Cauchy sequence is one where the terms all eventually get close to one another. The point of (6) is that these two notions are the same.

- For convenience we will also record the definition of when a sequence diverges to ∞, which we will use occasionally later:
- Definition: We say a sequence $\{a_n\}_{n\geq 1}$ of complex numbers diverges to ∞ if for any $A > 0$ there exists a positive integer N such that for all $n \geq N$ it is true that $|a_n| > A$.
  - The intuition here is that a sequence diverges to ∞ if the absolute values of the terms a_n eventually stay arbitrarily large as n grows large. (Unlike with sequences of real numbers, where we have separate notions of diverging to $+\infty$ and diverging to $-\infty$, we do not generally make a distinction about the "direction" in which the a_n tend toward ∞.)
  - Example: The sequence with $a_n = (1+i)^n$ diverges to ∞ as $n \to \infty$ since $|a_n| = \sqrt{2}^n$ tends to ∞ as n does.
  - Note that a non-convergent sequence need not diverge to ∞: for example, the sequence with $a_n = i^n$ does not diverge to ∞ since $|a_n| = 1$ for all n.
- Our main interest is in discussing convergence of infinite series of complex numbers. As usual we define the associated notions in the same way as for real series:
- Definition: If $\{a_n\}_{n\geq 1}$ is a sequence of complex numbers, for each integer $k \geq 1$ we define the kth partial sum of the series to be $S_k = \sum_{n=1}^{k} a_n = a_1 + \cdots + a_k$. We say the infinite series $\sum_{n=1}^{\infty} a_n$ converges if the limit of partial sums $\lim_{k\to\infty} \sum_{n=1}^{k} a_n$ converges, and in such a case we say the value of the series is the resulting limit.
  - Example: Suppose $|z| < 1$: we claim that the infinite geometric series $\sum_{n=1}^{\infty} z^{n-1}$ converges and has sum $\dfrac{1}{1-z}$. Its kth partial sum equals $1 + z + z^2 + \cdots + z^{k-1} = \dfrac{1-z^k}{1-z}$ by the usual finite geometric series summation formula, and so for $L = \dfrac{1}{1-z}$ we compute the difference $|S_k - L| = \left|\dfrac{z^k}{1-z}\right| = \dfrac{|z|^k}{|1-z|}$ which approaches 0 as $k \to \infty$ because $|z| < 1$. Thus, the series converges and its sum is $\dfrac{1}{1-z}$ as claimed.
  - Example: The infinite series $\sum_{n=1}^{\infty} \dfrac{1}{n^2+n}$ converges and has sum 1, since its kth partial sum equals $\sum_{n=1}^{k} \dfrac{1}{n^2+n} = \sum_{n=1}^{k} \left[\dfrac{1}{n} - \dfrac{1}{n+1}\right] = [\dfrac{1}{1} - \dfrac{1}{2}] + [\dfrac{1}{2} - \dfrac{1}{3}] + \cdots + [\dfrac{1}{k} - \dfrac{1}{k+1}] = 1 - \dfrac{1}{k+1}$ which has limit 1 as $k \to \infty$.
- We have various fundamental properties of convergent series:
- Proposition (Properties of Series Convergence): Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are complex series.
  1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n$ exists and is 0.
     - Proof: Note by hypothesis the sequence of partial sums $\{S_n\}_{n\geq 1}$ is a Cauchy sequence, which in particular requires the differences $S_n - S_{n-1}$ between consecutive terms to tend to 0.
     - But since $S_n - S_{n-1} = a_n$ this means $\lim_{n\to\infty} a_n = 0$ as claimed.

2. (Linearity) If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ both converge, then $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$ and $\sum_{n=1}^{\infty} ca_n = cA$ for any $c \in \mathbb{C}$.
3. The series $\sum_{n=1}^{\infty} a_n$ converges to a sum $x+iy$ if and only if $\sum_{n=1}^{\infty} \mathrm{Re}(a_n)$ converges to x and $\sum_{n=1}^{\infty} \mathrm{Im}(a_n)$ converges to y.
   - Proofs: Both (2) and (3) follow immediately from the corresponding properties of limits of sequences.
4. (Absolute Convergence) If $\sum_{n=1}^{\infty} |a_n|$ converges (in such a case we say the series converges absolutely), then $\sum_{n=1}^{\infty} a_n$ converges.
   - Proof: Let $\epsilon > 0$. Since the partial sums $\tilde{S}_k = \sum_{n=1}^{k} |a_n|$ are Cauchy, there exists N such that $\left|\tilde{S}_m - \tilde{S}_n\right| < \epsilon$ for all $m > n \geq N$.
   - Then for $S_k = \sum_{n=1}^{k} a_n$, by the triangle inequality we have $|S_m - S_n| = |a_{n+1} + \cdots + a_m| \leq |a_{n+1}| + \cdots + |a_m| = \tilde{S}_m - \tilde{S}_n < \epsilon$, and so the partial sums of $\{a_n\}_{n\geq 1}$ are also Cauchy.

2.2.2 Convergent Power Series

- Our primary goal is to construct functions as power series, so in order to connect our previous discussion of formal power series to power series as functions, we need to analyze the convergence of power series.
  - Explicitly, given a formal power series $f = \sum_{n=0}^{\infty} a_n z^n$ and a value $z_0 \in \mathbb{C}$, we would now like to (attempt to) "plug in z_0" to the formal power series.
  - Of course, the resulting series may not converge. If we denote by S the set of points where the series does converge, we may then view f as a function $f : S \to \mathbb{C}$ defined by the series expansion $f(z_0) = \sum_{n=0}^{\infty} a_n z_0^n$.
  - All of this works just as well if f is a formal Laurent series, except for the fact that a formal Laurent series of negative order will not be defined when $z_0 = 0$ since the resulting series expansion would require dividing by zero. (For example, what would $z^{-1} + 2$ mean when $z = 0$?)
  - However, since a Laurent series only involves finitely many terms of negative order, the convergence will be unaffected for $z_0 \neq 0$ if we discard those terms, so we really only need to analyze convergence of power series.

- Our main result is that every power series has an associated disc of convergence, inside which the series converges absolutely and outside which the series diverges.

- Proposition (Convergent Power Series): Suppose $\sum_{n=0}^{\infty} a_n z^n$ is a formal power series and let $z_0 \in \mathbb{C}$.
  1. If $\sum_{n=0}^{\infty} a_n z^n$ converges for $z = z_0$, then it converges absolutely for all z_1 with $|z_1| < |z_0|$.
     - Proof: If $z_0 = 0$ the result is trivial so assume $|z_0|$ is positive and that $|z_1| < |z_0|$. Since the series $\sum_{n=0}^{\infty} a_n z_0^n$ converges, the individual terms must tend to zero, meaning that $\lim_{n\to\infty} a_n z_0^n = 0$.
     - By the definition of limit with $\epsilon = 1$, there exists some N such that for all $n \geq N$ we have $|a_n z_0^n| \leq 1$, so that and thus $|a_n| \leq |z_0|^{-n}$.
     - But then $\displaystyle\sum_{n=N}^{\infty} |a_n z_1^n| \leq \sum_{n=N}^{\infty} \left|\frac{z_1}{z_0}\right|^n$, and this last series is a convergent geometric series because $\left|\dfrac{z_1}{z_0}\right| < 1$. This implies the original series converges absolutely, as claimed.
  2. There exists a nonnegative number R (which can equal ∞), the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$, such that the series converges absolutely for $|z| < R$ and diverges for $|z| > R$.
     - Proof: Consider the set of nonnegative real numbers $S = \{|z| : \sum_{n=0}^{\infty} a_n z^n \text{ converges}\}$. Note $0 \in S$ so S is nonempty.
     - If S has no upper bound, then by (1) applied to an increasing sequence of values of $|z|$, the series converges absolutely for all $z \in \mathbb{C}$, in which case $R = \infty$.
     - Otherwise, S is bounded above. By the least upper bound axiom of the real numbers[2] S has some least upper bound R. If $R = 0$, then the series converges only when $z = 0$.

[2]We say a real number b is an upper bound for a set S of real numbers if $x \leq b$ for all $x \in S$. A least upper bound for a set S is a number l that is an upper bound for S such that $l \leq b$ for any other upper bound of S. The least upper bound axiom states that if S is a nonempty subset of $\mathbb{R}$ that has an upper bound, then S has a least upper bound.

- Otherwise, if $R > 0$, then for any $\epsilon > 0$ there exists some $r \in S$ with $R - \epsilon < r \le R$ such that the series converges for some z_0 with $|z_0| = r$ (otherwise, $R - \epsilon$ would be a least upper bound for S). Then by (1), the series converges absolutely for all $|z_1| < R - \epsilon < r$. Since this holds for any $\epsilon > 0$ we conclude that the series converges absolutely for all $|z| < R$ as claimed.
- Finally, by the definition of R, the series will diverge whenever $|z| > R$: otherwise, R would not be the least upper bound of the set of values of $|z|$ where the series converges.

3. If the series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, then $\limsup_{n\to\infty} |a_n|^{1/n} \le 1/R$.
   - Note that if $\{b_n\}$ is any sequence, then $\limsup_{n\to\infty} b_n$ is an abbreviation for $\lim_{n\to\infty}[\sup_{k\ge n} b_k]$ where $\sup_{k\ge n} b_k$ is the least upper bound of the set $\{b_n, b_{n+1}, b_{n+2}, \dots\}$.
   - The least upper bound is either finite (if the sequence is bounded) or ∞ (if not). The limit supremum given above is, roughly speaking, capturing the idea that if the values $|a_n|^{1/n}$ are bounded above, then the radius of convergence is positive and at least the reciprocal of the "limiting" least upper bound.
   - Proof: Suppose that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, and let $0 < r < R$.
   - By (2), the series $\sum_{n=0}^{\infty} |a_n| r^n$ converges, so its terms must go to zero.
   - This means $\lim_{n\to\infty} |a_n| r^n = 0$, and so by the definition of limit there exists some N such that $|a_n| r^n \le 1$ hence $|a_n|^{1/n} \le 1/r$ for all $n \ge N$.
   - This means $1/r$ is an upper bound for the set $\{|a_n|^{1/n}\}_{n\ge N}$ hence also for all of the sets $\{|a_n|^{1/n}\}_{n\ge M}$ for any $M \ge N$.
   - Therefore, $\limsup_{n\to\infty} |a_n|^{1/n} \le 1/r$. Since this holds for any $0 < r < R$ we conclude in fact that $\limsup_{n\to\infty} |a_n|^{1/n} \le 1/R$. This establishes the result when R is finite.
   - In the case $R = \infty$, we may apply the argument above to an increasing sequence of values of R to conclude that $\limsup_{n\to\infty} |a_n|^{1/n} = 0$.
4. If $\limsup_{n\to\infty} |a_n|^{1/n} = t$ is finite, then the series $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent for $|z| < 1/t$.
   - Proof: Suppose that $\limsup_{n\to\infty} |a_n|^{1/n} = t$ for some $t > 0$ and let $t'' > t' > t$.
   - By the definition of the limsup, there exists some N such that for all $n \ge N$ we have $|a_n|^{1/n} \le t'$ so that $|a_n| \le (t')^n$.
   - Then for $|z| < 1/t''$, the series tail $\sum_{n=N}^{\infty} |a_n z^n| \le \sum_{n=N}^{\infty} \left(\frac{t'}{t''}\right)^n$ is bounded by a convergent geometric series, and so $\sum_{n=N}^{\infty} a_n z^n$ and hence $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < 1/t''$.
   - Since this holds for any $t'' > t$, taking $t'' \to t$ shows that the series converges absolutely for all $|z| < 1/t$.
5. The radius of convergence R of $\sum_{n=0}^{\infty} a_n z^n$ is given by $R = 1/\limsup_{n\to\infty} |a_n|^{1/n}$ (where if the limsup is 0 then $R = \infty$ and if the limsup is ∞ then $R = 0$). In particular, if $\lim_{n\to\infty} |a_n|^{1/n}$ exists then the value of the limit is $1/R$.
   - We note that the second part is simply the standard root test applied to the series $\sum_{n=0}^{\infty} a_n z^n$. The point of this result is that we can also handle the situation where the limit required for the root test does not exist, but the terms $|a_n|^{1/n}$ are still bounded.
   - Proof: The first part follows by combining (3) and (4). The second part follows by noting that if $\lim_{n\to\infty} |a_n|^{1/n}$ exists then the value of the limit equals $\limsup_{n\to\infty} |a_n|^{1/n}$.

- We can use the explicit formula in (5) to compute the radius of convergence R of various power series.
  - Inside the radius of convergence the series converges absolutely and outside it will diverge (indeed, it will diverge to ∞, since the absolute values of the terms will not go to zero).
  - However, the behavior on the boundary circle $|z| = R$ can be quite complicated: the series may converge at some points but not others.
  - When computing the radius of convergence, it is often useful to use the fact that $\lim_{n\to\infty} n^{1/n} = 1$ along with the simple estimates $(n/e)^n < n! < n^n$, the upper bound following from noting that each term in $n!$ is at most n and the lower bound following from observing that $e^n = \sum_{k=0}^{\infty} \frac{k^n}{k!}$ is greater than its nth

term $\frac{n^n}{n!}$. One may also use the more precise Stirling's approximation $n! \approx n^n e^{-n}\sqrt{2\pi n}$, in which the ratio between the two quantities approaches 1 as $n \to \infty$.

- Example: Find the radius of convergence of $\sum_{n=0}^{\infty} z^n$ and all values of z for which it converges.
  - We have $a_n = 1$ for all n, so the limit $\lim_{n\to\infty} |a_n|^{1/n}$ exists and equals 1. Thus, the radius of convergence is $\boxed{1}$, meaning that the series converges for $|z| < 1$ and diverges for $|z| > 1$.
  - It remains to analyze the situation where $|z| = 1$. In this case we see that the terms all have $|z^n| = 1$, so since the terms do not have limit 0, the series does not converge for any of these values of z.
  - We conclude that the series converges for $|z| < 1$ and diverges for $|z| \geq 1$.

- Example: Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{n}{(2+i)^n} z^n$ and all values of z for which it converges.
  - We have $a_n = \frac{n}{(2+i)^n}$ for all n, so $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}}{\sqrt{5}} = \frac{1}{\sqrt{5}}$. Thus, the radius of convergence is $\boxed{\sqrt{5}}$, meaning that the series converges for $|z| < \sqrt{5}$ and diverges for $|z| > \sqrt{5}$.
  - When $|z| = \sqrt{5}$, the terms have $\left|\frac{n}{(2+i)^n} z^n\right| = n\frac{\sqrt{5}^n}{\sqrt{5}^n} = n$, so since the terms do not have limit 0, the series does not converge for any of these values of z.
  - We conclude that the series converges for $|z| < \sqrt{5}$ and diverges for $|z| \geq \sqrt{5}$.

- Example: Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{z^n}{2^n n^2}$ and all values of z for which it converges.
  - We have $a_n = \frac{1}{2^n n^2}$ for all n, so $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{1}{2}(n^{1/n})^{-2} = \frac{1}{2}$. Thus, the radius of convergence is $\boxed{2}$, meaning that the series converges for $|z| < 2$ and diverges for $|z| > 2$.
  - When $|z| = 2$, the terms have absolute value $\left|\frac{z^n}{2^n n^2}\right| = \frac{1}{n^2}$, so since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is finite (by comparison to the integral $\int_1^{\infty} \frac{dx}{x^2}$ or to the series $1 + \sum_{n=2}^{\infty} \frac{1}{n^2 - n} = 2$), the series converges absolutely for all $|z| = 2$.
  - We conclude that the series converges for $|z| \leq 2$ and diverges for $|z| > 2$.

- Example: Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ and all values of z for which it converges.
  - We have $a_n = \frac{1}{n!}$ for all n, so the limit $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{1}{(n!)^{1/n}} = 0$ since $(n!)^{1/n} > n/e$ for each n.
  - Thus, the radius of convergence is $\boxed{\infty}$, meaning that the series converges for all $z \in \mathbb{C}$.

- Example: Find the radius of convergence of $\sum_{n=0}^{\infty} n! z^n$ and all values of z for which it converges.
  - We have $a_n = n!$ for all n, so the limit $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} (n!)^{1/n} = \infty$ since $(n!)^{1/n} > n/e$ for each n.
  - Thus, the radius of convergence is $\boxed{0}$, meaning that the series converges only for $z = 0$.

- There are many examples of series with more unusual formulas for the coefficients, and their behaviors can be a bit more exotic:

- Example: Analyze the convergence of the power series $\sum_{n=0}^{\infty} \left[\frac{z^{2n}}{2^n} + \frac{z^{2n+1}}{3^n}\right] = 1 + z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{9} + \frac{z^6}{8} + \frac{z^7}{27} + \cdots$.

- Note that the limit $\lim_{n\to\infty} |a_n|^{1/n}$ does not exist, since $|a_n|^{1/n} = \dfrac{1}{\sqrt{2}}$ when n is even but $|a_n|^{1/n} = \dfrac{1}{3^{n/(2n+1)}} \to \dfrac{1}{\sqrt{3}}$ when n is odd.
- However, the limsup still exists and is $\dfrac{1}{\sqrt{2}}$ because $\sup_{k\geq n} |a_k|^{1/k} = \dfrac{1}{\sqrt{2}}$ for all $k \geq 1$, and so $\lim_{n\to\infty} \sup_{k\geq n} |a_k|^{1/k} = \dfrac{1}{\sqrt{2}}$.
- Thus, the radius of convergence is $\boxed{\sqrt{2}}$, so the series converges absolutely for $|z| < \sqrt{2}$ and diverges for $|z| > \sqrt{2}$.
- Remark: The idea here is that this series alternates taking terms from the series $\sum_{n=0}^{\infty} \dfrac{z^{2n}}{2^n}$ (which has radius of convergence $\sqrt{2}$) with taking terms from the series $\sum_{n=0}^{\infty} \dfrac{z^{2n+1}}{3^n}$ (which has radius of convergence $\sqrt{3}$). Its overall behavior is controlled by the series with the smaller radius of convergence.

- Example: Analyze the convergence of the power series $\sum_{n=1}^{\infty} \dfrac{z^{2^n}}{n} = z^2 + \dfrac{z^4}{2} + \dfrac{z^8}{3} + \cdots$.
  - Note that the limit $\lim_{n\to\infty} |a_n|^{1/n}$ does not exist, since $|a_n|^{1/n} = n^{-1/2^n}$ when n is a power of 2 and $|a_n|^{1/n} = 0$ otherwise. However, the limsup still exists since we may simply ignore the zero terms and compute $\lim_{n\to\infty} n^{-1/2^n} = 1$.
  - Thus, the radius of convergence is 1, so the series converges absolutely for $|z| < 1$ and diverges for $|z| > 1$.
  - The convergence behavior of this series on the unit circle $|z| = 1$ turns out to be rather erratic. If $z = e^{2\pi a i/2^k}$ is a 2^kth root of unity, then $z^{2^k} = 1$ and so all of the higher 2^nth powers of z will also be 1. Then the tail of the series $\sum_{n=1}^{\infty} \dfrac{z^{2^n}}{n}$ is $\sum_{n=k}^{\infty} \dfrac{1}{n}$, which diverges to ∞.
  - Thus, we see that the series diverges on all of the points of the form $z = e^{2\pi a i/2^k}$ for some $k \geq 1$. This set of points is dense on the unit circle, in the sense that any point on the circle is a limit along a sequence of such points (this follows simply because we may take a sequence of rational numbers of the form $a_k/2^k$ converging to an arbitrary number θ: then $e^{2\pi i a_k i/2^k} \to e^{i\theta}$)
  - On the other hand, it can also be shown using Fourier analysis[3] that there is a dense set of points on the unit circle for which the series converges.

- We can also describe how the various algebraic operations on power series affect their convergence:
- Proposition (Operations on Power Series): Suppose that $f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$ are power series whose radii of convergence are both at least $R > 0$.

  1. For $|z| < R$, the series for cf for any $c \in \mathbb{R}$, and for $f + g$ and $f - g$, all converge, and $(cf)(z) = c \cdot f(z)$, $(f + g)(z) = f(z) + g(z)$, and $(f - g)(z) = f(z) - g(z)$.
     - Proof: For cf we have $\lim_{n\to\infty} \sum_{k=0}^{\infty}(ca_k)z^k = \lim_{n\to\infty}[c \cdot \sum_{k=0}^{\infty} a_k z^k] = c \cdot \lim_{n\to\infty}[\sum_{k=0}^{\infty} a_k z^k] = c \cdot f(z)$ since the latter limit exists.
     - So we see that $cf(z)$ converges to the claimed value $c \cdot f(z)$. Likewise, for $f + g$ and $f - g$ we simply apply the corresponding limit properties to see they converge to their claimed values.
  2. For $|z| < R$, the series for fg converges and $(fg)(z) = f(z)g(z)$.
     - Proof: As a formal power series we have $fg = \sum_{n=0}^{\infty} c_n z^n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Let r be arbitrary with $0 < r < R$.
     - From our characterization of the radius of convergence, we know that $\limsup_{n\to\infty} |a_n|^{1/n} \geq 1/R > 1/r$ so there exists N_1 such that for all $n \geq N_1$ we have $|a_n|^{1/n} \leq 1/r$ meaning that $|a_n| \leq 1/r^n$. Likewise since $\limsup_{n\to\infty} |b_n|^{1/n} \geq 1/r$ there exists N_2 such that for all $n \geq N_2$ we have $|b_n| \leq 1/r^n$.

[3] Specifically, Carleson's theorem (applied to an appropriate function f) implies that if $\sum_{n=0}^{\infty} a_n^2$ converges then $\sum_{n=0}^{\infty} a_n e^{inx}$ converges for almost all x in $[0, 2\pi]$, in the sense that the Lebesgue measure of the set of points where the series diverges is equal to 0.

- Taking $N = \max(N_1, N_2)$ yields that $|a_n|, |b_n| \le 1/r^n$ for all $n \ge N$. By accounting for the behavior of the lower terms, this means there exist constants A and B such that $|a_n| \le A/r^n$ and $|b_n| \le B/r^n$ for all n.
- Then $|c_n| = |\sum_{k=0}^n a_k b_{n-k}| \le \sum_{k=0}^n |a_k| |b_{n-k}| \le \sum_{k=0}^n (AB/r^n) = (n+1)AB/r^n$, and so $|c_n|^{1/n} \le (n+1)^{1/n} A^{1/n} B^{1/n}/r$.
- Taking the limsup yields $\limsup_{n\to\infty} |c_n|^{1/n} \le 1/r$, whence the radius of convergence of fg is at least r. Since this holds for any $0 < r < R$ the radius of convergence of fg is at least R.
- As a consequence we also see that $\sum_{n=0}^\infty [\sum_{k=0}^n |a_k| |b_{n-k}|] r^n$ converges.
- For the actual value, let $f_n(z) = \sum_{k=0}^n a_n z^n$, $g_n(z) = \sum_{k=0}^n b_n z^n$, and $p_n(z) = \sum_{k=0}^n c_n z^n$. Then the only terms in $f_n(z) g_n(z) - p_n(z)$ are those of order at least $n+1$.
- Then $\lim_{n\to\infty} |f_n(z) g_n(z) - p_n(z)| \le \lim_{n\to\infty} \sum_{j=n+1}^\infty [\sum_{k=0}^n |a_k| |b_{j-k}|] |z|^n$, but this limit converges to 0 since it is the tail of the convergent series $\sum_{n=0}^\infty [\sum_{k=0}^n |a_k| |b_{n-k}|] r^n$ from above.
- This implies $\lim_{n\to\infty} [f_n(z) g_n(z) - p_n(z)] = 0$, hence $(fg)(z) = \lim_{n\to\infty} p_n(z) = \lim_{n\to\infty} f_n(z) \cdot \lim_{n\to\infty} g_n(z) = f(z)g(z)$ as desired.

3. If f has nonzero constant term, then the radius of convergence for f^{-1} is positive.
   - Proof: By rescaling we may assume that f has constant term 1.
   - Now, since $f = \sum_{n=0}^\infty a_n z^n$ has a positive radius of convergence, by our results above there exists a finite constant A such that $|a_n|^{1/n} \le A$ for all sufficiently large n. By increasing A as needed, we may in fact assume $|a_n|^{1/n} \le A$ hence $|a_n| \le A^n$ for all $n \ge 1$.
   - Then $f^{-1} = \sum_{n=0}^\infty b_n z^n$ where $b_0 = 1$ and $b_n = -\sum_{k=1}^n a_k b_{n-k}$ for each $n \ge 1$.
   - Hence, $|b_n| = |\sum_{k=1}^n a_k b_{n-k}| \le \sum_{k=1}^n |a_k| |b_{n-k}| = \sum_{k=1}^n A^k |b_{n-k}|$.
   - Now we show that $|b_n| \le 2^{n-1} A^n$ for each $n > 0$ by strong induction on n. The base case $n = 1$ is trivial since $|b_1| \le A$.
   - For the inductive step suppose $|b_k| \le 2^{k-1} A^k$ for each $k \le n$. Then $|b_{n+1}| \le A|b_n| + A^2 |b_{n-1}| + \cdots + A^{n+1} |b_0| = A \cdot 2^{n-1} A^n + A^2 \cdot 2^{n-2} A^{n-1} + \cdots + A^{n+1} = (2^{n-1} + 2^{n-2} + \cdots + 1) A^{n+1} < 2^n A^{n+1}$ as required.
   - Finally, we see that for $|z| < 1/(3A)$, we have $\sum_{n=0}^\infty |b_n z^n| < \sum_{n=0}^\infty (3A)^{-n} (2^n A^{n+1}) = \sum_{n=0}^\infty A(2/3)^n = 3A$. Thus the series for f^{-1} converges absolutely for $|z| < 1/(3A)$, and thus has a positive radius of convergence.

- In general the radius of convergence of f^{-1} can be much smaller (or larger) than that of f and cannot be determined using only the radius of convergence of f.
  - For example, if $r > 0$ then for $f = 1 - rz$ the radius of convergence is ∞, but for $f^{-1} = \sum_{n=0}^\infty r^n z^n$ the radius of convergence is $1/r$, which can be arbitrarily close to 0.
  - On the other hand, for $f = \sum_{n=0}^\infty (n+1) z^n$ the radius of convergence is 1 but for $f^{-1} = 1 - 2z + z^2$ the radius of convergence is ∞.
  - However, for the slightly different $f = \sum_{n=0}^\infty (n+2) z^n$ the radius of convergence is still 1, but now $f^{-1} = -z + \sum_{n=0}^\infty \dfrac{z^n}{2^{n+1}}$ has radius of convergence 2.

2.2.3 Continuity and Differentiability of Power Series

- We now investigate continuity and differentiability of power series. Our main goal is to show that a power series defines a differentiable function inside its radius of convergence, which in particular allowing us to construct holomorphic functions as power series. To establish these results, however, we require some preliminary facts about uniform convergence of functions.
  - To give some brief motivation, suppose that $\{f_n\}_{n\ge1}$ is a sequence of functions on a set S that converges pointwise as $n \to \infty$, meaning that $\lim_{n\to\infty} f_n(z)$ converges for each $z \in S$.
  - Then, even if each of the functions f_n is continuous, the resulting limit function $f(z) = \lim_{n\to\infty} f_n(z)$ need not be continuous: for example, if we take $f_n(x) = x^n$ on the interval $[0,1]$, then $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \end{cases}$ is discontinuous.

- The issue, broadly speaking, is that the functions f_n do not converge uniformly to f, in the sense that the maximum value of $|f_n(x) - f(x)|$ for $x \in [0,1]$ does not tend to 0 as $n \to \infty$.

- Definition: If S is a set of complex numbers, a sequence of complex functions $\{f_n\}_{n\geq 1}$ converges uniformly to f on S if for any $\epsilon > 0$ there exists N such that $|f_n(z) - f(z)| < \epsilon$ for all $z \in S$ and all $n \geq N$. We also say a series $\sum_{n=0}^{\infty} f_n$ converges uniformly if its partial sums converge uniformly.
  - Notice that the definition is more restrictive than requiring mere pointwise convergence, whose definition is as follows: for any $\epsilon > 0$ and any $z \in S$ there exists N such that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq N$.
  - The difference is that uniform convergence requires (for any $\epsilon > 0$) giving a single uniform choice of N that works for all $z \in S$ at once, rather than allowing different values of N for different points $z \in S$.
  - In essence, the idea is that allowing different values of N for different points $z \in S$ allows the convergence to vary in speed in S, to the extent that we can lose continuity after passing to the limit function f.
  - In contrast, insisting on uniform convergence turns out to force the limit function f to be continuous:

- Proposition (Uniform Convergence of Functions): Suppose R is a complex region and the complex functions $\{f_n\}_{n\geq 1}$ are continuous on R.
  1. Suppose that the sequence $\{f_n\}_{n\geq 1}$ converges uniformly to the function f on R. Then f is continuous on R.
     - Proof: Let $\alpha \in R$. We must show that $\lim_{z\to\alpha} f(z) = f(\alpha)$, so let $\epsilon > 0$. We must find $\delta > 0$ such that $|f(z) - f(\alpha)| < \epsilon$ for all $|z - \alpha| < \delta$.
     - By hypothesis, the functions f_n converge uniformly to f on R, so there exists N such that $|f_N(z) - f(z)| < \epsilon/3$ for all $z \in R$. In particular this holds for $z = \alpha$ so we also have $|f_N(\alpha) - f(\alpha)| < \epsilon/3$.
     - Also, since f_N is continuous, there exists $\delta > 0$ such that $|f_N(z) - f_N(\alpha)| < \epsilon/3$ for all z with $|z - \alpha| < \delta$.
     - Then by the triangle inequality, for all z with $|z - \alpha| < \delta$ we have $|f(z) - f(\alpha)| \leq |f(z) - f_N(z)| + |f_N(z) - f_N(\alpha)| + |f_N(\alpha) - f(\alpha)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$, as required.
     - Remark: This approach is a fairly common technique and is frequently called an "$\epsilon/3$ argument".
  2. Suppose that the sequence $\{f_n\}_{n\geq 1}$ is Cauchy, meaning that for any $\epsilon > 0$ there exists N such that $|f_n(z) - f_m(z)| < \epsilon$ for all $m, n \geq N$ and all $z \in R$. Then $\{f_n\}_{n\geq 1}$ converges uniformly to a function f.
     - Proof: First note that for any fixed $z \in R$, the sequence $\{f_n(z)\}_{n\geq 1}$ is Cauchy, so it converges to a limit $f(z)$. We now show that the convergence of $\{f_n\}_{n\geq 1}$ to the resulting function f is uniform.
     - Let $\epsilon > 0$. Then by hypothesis there exists N such that $|f_n(z) - f_m(z)| < \epsilon/2$ for all $m, n \geq N$ and all $z \in R$.
     - For a fixed $n \geq N$ and $z \in R$, because $\lim_{m\to\infty} f_m(z) = f(z)$ there exists an m (depending on z, n, ϵ) such that $|f_m(z) - f(z)| < \epsilon/2$.
     - Then by the triangle inequality, we have $|f(z) - f_n(z)| \leq |f(z) - f_m(z)| + |f_m(z) - f_n(z)| < \epsilon/2 + \epsilon/2 = \epsilon$. Since this holds for any ϵ and any $z \in R$, the functions $\{f_n\}_{n\geq 1}$ converge uniformly to f as required.
  3. If $\{f_n\}_{n\geq 1}$ is Cauchy, then $\{f_n\}_{n\geq 1}$ converges to a continuous function f on R.
     - Proof: This is immediate from (1) and (2).

- We can now apply these results to establish that power series are continuous inside of their radius of convergence, and also that a function is uniquely determined by its power series:

- Theorem (Continuity and Uniqueness of Power Series): Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence $R > 0$.
  1. For any $0 < r < R$, the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges absolutely and uniformly on the region $|z| \leq r$.
     - Proof: We have previously shown that if f has radius of convergence R, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R$, so in particular it converges absolutely on that region.

- For the uniformity, we will show that the sequence of partial sums $f_n = \sum_{k=0}^{n} a_k z^k$ is Cauchy on $|z| \le r$. Suppose for convenience that $n \le m$: then for $|z| \le r$ we have $|f_m(z) - f_n(z)| = \left|\sum_{k=n+1}^{m} a_k z^k\right| \le \sum_{k=n+1}^{m} \left|a_k z^k\right| \le \sum_{k=n+1}^{\infty} |a_k| \, r^k$.
- But because $f(z)$ converges absolutely at $z = r$ (since $r < R$), the tail sum $\sum_{k=n+1}^{\infty} |a_k| \, r^k$ tends to zero as $n \to \infty$, which is to say, there exists some N such that $\sum_{k=N+1}^{\infty} |a_k| \, r^k < \epsilon$.
- Then for $N \le n \le m$ we have $|f_m(z) - f_n(z)| < \epsilon$ for all $|z| \le r$, so the sequence is Cauchy hence converges uniformly by our results above.
- Remark: We will note that the series for f need not converge uniformly on $|z| < R$: for example, $f(z) = \sum_{n=0}^{\infty} z^n$ has radius of convergence 1, but the convergence is not uniform on $|z| < 1$ since $f(z) \to \infty$ as $z \to 1$ (so there is no uniform upper bound on the tails of the series as $z \to 1$).

2. (Continuity) The function $f(z)$ is continuous for all $|z| < R$.
   - Proof: Take any r with $|z| < r < R$. Then z lies in the region $|z| \le r$ and so f is continuous at z by (1) and the fact that a uniformly-convergent limit of continuous functions is continuous.
3. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a nonzero power series with $f(0) = 0$, then there exists some $s > 0$ such that $f(z) \ne 0$ for all z with $0 < |z| < s$.
   - The point of this result is that the zero at $z = 0$ of the power series $f(z)$ is "isolated": namely, that there are no other zeroes of f within some positive distance of $z = 0$.
   - Equivalently, by taking the contrapositive, if $f(z)$ is a convergent power series with a sequence $\{z_n\}_{n \ge 1}$ with $f(z_n) = 0$ and $z_n \ne 0$ for each n such that $z_n \to 0$, then in fact $f(z)$ must be the zero series.
   - Proof: Suppose f has order $d \ge 1$, so that $f(z) = a_d z^d + \sum_{n=d+1}^{\infty} a_n z^n = a_d z^d [1 + \sum_{n=1}^{\infty} b_n z^n]$ where $b_n = a_{d+n}/a_d$ and $a_d \ne 0$.
   - Then the series $g(z) = \sum_{n=1}^{\infty} b_n z^n$ has the same radius of convergence $R > 0$, so by (2) it defines a continuous function with $g(0) = 1$ (since its constant term is 1). In particular, there exists some $s > 0$ such that $|g(z)| > 1/2$ for all $|z| < s$.
   - But then for $0 < |z| < s$ the function $f(z) = a_d z^d g(z)$ is nonzero since none of a_d, z^d, and $g(z)$ is zero.
4. (Uniqueness) If $f(z)$ and $g(z)$ are power series with radii of convergence $\ge R$, and $f(z_n) = g(z_n)$ for an infinite sequence $\{z_n\}_{n \ge 1}$ of nonzero complex numbers with limit 0, then in fact $f(z)$ and $g(z)$ are equal as power series.
   - Proof: Suppose $f(z) \ne g(z)$ and apply (3) to $f(z) - g(z)$: then there exists some $s > 0$ such that $f(z) - g(z) \ne 0$ for all z with $0 < |z| < s$.
   - But this is directly contradicted by the assumption that the limit $z_n \to 0$, since there must exist terms z_n with $0 < |z_n| < s$.
   - Therefore, $f(z) = g(z)$ as power series.

- Finally, we tackle differentiation and antidifferentiation of power series: we will show that we can calculate derivatives and antiderivatives by differentiating and antidifferentiating termwise.
- Theorem (Derivatives and Antiderivatives of Power Series): Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence $R > 0$.

1. The termwise derivative series $\sum_{n=0}^{\infty} n a_n z^{n-1}$ has radius of convergence R.
   - Proof: By our results on the radius of convergence we know that $\limsup_{n\to\infty} |a_n|^{1/n} = 1/R$.
   - But then since $\lim_{n\to\infty} |n|^{1/n} = 1$, we also have $\limsup_{n\to\infty} |n a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n} = 1/R$, so the series $\sum_{n=0}^{\infty} n a_n z^{n-1}$ also has radius of convergence R.
2. The function $f(z)$ is holomorphic on the region $|z| < R$ and its derivative is $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$.
   - Proof: Suppose $|z| < R$ and let $\delta > 0$ be such that $|z| + \delta < R$ (e.g., $\delta = (R - |z|)/2$).
   - In computing the limit of the difference quotient $\lim_{h\to 0} \dfrac{f(z+h) - f(z)}{h}$ we may assume $|h| < \delta$ so that f converges absolutely at z and at $z + h$.

- For such h we have $f(z+h) = \sum_{n=0}^{\infty} a_n(z+h)^n = \sum_{n=0}^{\infty} a_n(z^n + nz^{n-1}h + h^2 p_n(z,h))$ where $p_n(z,h) = \sum_{k=2}^{n} \binom{n}{k} h^{k-2} z^{k-h}$.
- We have $|p_n(z,h)| = \left|\sum_{k=2}^{n} \binom{n}{k} h^{k-2} n^{k-h}\right| \le \sum_{k=2}^{n} \binom{n}{k} |h|^{k-2} |z|^{k-h} \le \sum_{k=2}^{n} \binom{n}{k} \delta^{k-2} |z|^{n-k} = p_n(|z|, \delta)$.
- Therefore we have $f(z+h) - f(z) - h\sum_{n=0}^{\infty} na_n z^{n-1} = h^2 \sum_{n=0}^{\infty} a_n p_n(z,h)$.
- Since the three series on the left-hand side all converge absolutely (using (1) for $\sum_{n=0}^{\infty} na_n z^{n-1}$), the series $\sum_{n=0}^{\infty} a_n p_n(z,h)$ on the right-hand side also converges absolutely. In particular this means $\sum_{n=0}^{\infty} |a_n| \, p_n(|z|, \delta)$ is some finite number M.
- Then $\left|\dfrac{f(z+h)-f(z)}{h} - \sum_{n=0}^{\infty} na_n z^{n-1}\right| = |h| \cdot |\sum_{n=0}^{\infty} p_n(z,h)| \le |h| \cdot \sum_{n=0}^{\infty} |a_n| \, |p_n(z,h)| \le |h| \cdot \sum_{n=0}^{\infty} a_n p_n(|z|, \delta) = M\,|h|$.
- But now since $M\,|h| \to 0$ as $h \to 0$, we see that the limit $\lim_{h\to 0} \left[\dfrac{f(z+h)-f(z)}{h} - \sum_{n=0}^{\infty} na_n z^{n-1}\right]$ exists and equals zero.
- This means $f'(z)$ exists and $f'(z) = \lim_{h\to 0} \dfrac{f(z+h)-f(z)}{h} = \sum_{n=0}^{\infty} na_n z^{n-1}$, as claimed.

3. The function $f(z)$ is infinitely differentiable and its higher derivatives $f^{(n)}$ all have radius of convergence R. Furthermore, the values of the derivatives at $z = 0$ determine the coefficients a_n via $f^{(n)}(0) = n!a_n$.
   - Proof: The first statement follows by a trivial induction using (2). For the coefficients, we simply observe that the constant term of the series for $f^{(n)}$ is obtained by differentiating the term $a_n z^n$ a total of n times, yielding $n!a_n$.
4. The termwise antiderivative series $\sum_{n=0}^{\infty} \dfrac{a_n}{n+1} z^{n+1}$ has radius of convergence R.
   - Proof: As with the derivative series, since $\limsup_{n\to\infty} |a_n|^{1/n} = 1/R$ and since $\lim_{n\to\infty} \left|\dfrac{1}{n+1}\right|^{1/n} = 1$, we have $\limsup_{n\to\infty} \left|\dfrac{a_n}{n+1}\right|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n} = 1/R$, so the series $\sum_{n=0}^{\infty} \dfrac{a_n}{n+1} z^{n+1}$ also has radius of convergence R.
5. The function $f(z)$ is antidifferentiable on $|z| < R$ with an antiderivative given by $F(z) = \sum_{n=0}^{\infty} \dfrac{a_n}{n+1} z^{n+1}$.
   - Proof: The termwise antiderivative series $F(z)$ is holomorphic for $|z| < R$ by (2) and (3) and its derivative is easily seen to be $f(z)$ by the differentiation formula in (2).

2.2.4 Analytic Functions and Power Series

- So far, we have only discussed power series of the form $\sum_{n=0}^{\infty} a_n z^n$, which will converge on an open disc of the form $|z| < R$ where R is the radius of convergence: we can think of this series as being centered at the (actual) center $z = 0$ of the open disc.
  - However, just as with Taylor series for real-valued functions, many complex functions are more naturally expressed as a power series centered at some other point $z = z_0$ rather than $z = 0$.
  - We can easily adapt our analysis so far by making the simple translation $z \mapsto z - z_0$ (thus shifting z_0 to 0) to obtain power series of the form $f = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ centered at $z = z_0$.
- All of our analysis so far also applies to these recentered power series: for example, the power series $f = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ possesses a radius of convergence R satisfying $1/R = \limsup_{n\to\infty} |a_n|^{1/n}$ with the property that f converges absolutely on the open disc $|z - z_0| < R$ and diverges for $|z - z_0| > R$.
  - In other words, we can calculate the radius of convergence in exactly the same manner as for a series centered at zero.
  - Example: The power series $\sum_{n=0}^{\infty} (z-1)^n$ centered at $z = 1$ has radius of convergence 1, so it converges absolutely for $|z-1| < 1$ and diverges for $|z-1| > 1$.

- Example: The power series $\sum_{n=0}^{\infty} \dfrac{(z-3i)^n}{2^n}$ centered at $z = 3i$ has radius of convergence 2, so it converges absolutely for $|z-3i| < 2$ and diverges for $|z-3i| > 2$.
- Example: The power series $\sum_{n=0}^{\infty} \dfrac{(z+i)^n}{n!}$ centered at $z = i$ has radius of convergence ∞, so it converges absolutely on all of $\mathbb{C}$.

- We can make all of this precise as follows:
- Definition: If f is a complex function, we say f is analytic at z_0 if there exists a power series expansion $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and some $r > 0$ such that the series converges to the value $f(z)$ for all z with $|z - z_0| < r$. If U is an open region, we say f is analytic on U if f is analytic at every point in U.
  - In other words, an analytic function is one that can be written as an absolutely convergent power series centered around any point z_0 in its domain. We think of f as being defined "locally", i.e., near z_0, by that power series.
  - Example: Polynomials are analytic on every open set U, since we may explicitly change variables to obtain the desired "series" expansion.
  - If f is analytic on U, then each of the power series expansions at $z_0 \in U$ are uniquely determined: if we had two different series expansions, their difference would be zero on an open disc centered at z_0, but this requires the series to be identical by our uniqueness result earlier.
  - We can see immediately from our results on convergence that if f and g are analytic on U, then so are $f+g$, $f-g$, fg, and that f/g is analytic on the subset of U where $g(z) \neq 0$.
  - As a consequence, we see that rational functions are also analytic, since they are quotients of polynomials.
- Furthermore, we see from our results on derivatives that if f is analytic on U, then f is holomorphic on U.
  - In the next chapter, we will prove that the converse is also true: if f is holomorphic on U, then f is analytic on U.
  - Interestingly, the analogous statement (differentiable implies analytic) is *false* for real-valued functions on the real line, even if we strengthen the hypothesis to "infinitely differentiable"!
  - For example, the function $f(x) = e^{-1/x^2}$ for $x \neq 0$ with $f(0) = 0$ is infinitely differentiable at 0 (all its derivatives are zero) but it is not analytic at 0 since it does not equal its power series expansion (the zero series) on any open interval centered at 0.
- Although we have defined analyticity, we have not actually shown yet that any particular power series are analytic other than rational functions.
  - To do this, we need to describe how to "recenter" a power series centered at some point $z = z_0$ as a series centered at a different point $z = z_1$.
  - A natural way to try to do this is to expand out the original series in terms of the new variable $z - z_1$ and then collect terms.
  - For simplicity, suppose that we want to rewrite $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ as a series centered at $z = 0$ (the general case can be obtained by a translation from this one).
  - The obvious approach is simply to expand out all of the powers $(z-z_0)^n$ formally and then add up the resulting partial sums, but this rapidly becomes messy, since each of the resulting coefficients ends up being an infinite sum.
  - A more efficient approach is to use the formula for the coefficients of a series in terms of the values of the derivative: with $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ we compute $f^{(k)}(z) = \sum_{n=0}^{\infty} a_n n(n-1)\cdots(n-k)(z-z_0)^{n-k-1}$ whence $f^{(k)}(0) = \sum_{n=0}^{\infty} a_n n(n-1)\cdots(n-k)(-z_0)^{n-k-1}$.
  - In particular, we can see, for example, that the constant term in the resulting series is $f(0) = \sum_{n=0}^{\infty} a_n(-z_0)^n$, which only makes sense when $f(0)$ converges.
  - However, the calculations above do presuppose that the desired series expansion actually exists, so we will need to show that as well. But in fact, as long as we insist that $z = 0$ lies inside the disc of convergence for $f(z)$, we can show that these formal calculations are valid.

- Proposition (Local Series Expansions): Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence R. Then f is analytic on the region with $|z| < R$.
  - By translating we also see that a power series $\sum_{n=0}^{\infty} a_n (z - z_1)^n$ with radius of convergence R is analytic for $|z - z_1| < R$.
  - Proof: Let z_0 be a point in the disc, so that $|z_0| < R$, and choose $s > 0$ so that $|z_0| + s < R$: we will construct a power series expansion $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ that converges to $f(z)$ for $|z - z_0| < s$.
  - We have $z^n = [z_0 + (z - z_0)^n] = \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} (z - z_0)^k$ so $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n \left[\sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} (z - z_0)^k\right]$.
  - Since f converges absolutely for all $|z| < R$ (and so in particular for all $|z - z_0| < s$ since this smaller disc is contained completely inside $|z| < R$) taking $\tilde{z} = |z| + |z - z_0|$ in the absolute value series, we see that the series $\sum_{n=0}^{\infty} |a_n| (|z| + |z - z_0|)^n = \sum_{n=0}^{\infty} |a_n| \left[\sum_{k=0}^{n} \binom{n}{k} |z_0|^{n-k} |z - z_0|^k\right]$ converges.
  - But this means the expression $\sum_{n=0}^{\infty} a_n \left[\sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} (z - z_0)^k\right]$ converges absolutely, and so we may switch the order of summation without changing the value.
  - This yields $f(z) = \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k}\right] (z - z_0)^k$ and so $f(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$ for $b_k = \sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k}$.

- In the proof above, note that the coefficient b_k is simply the value $\dfrac{f^{(k)}(z_0)}{k!}$, and so we have simply rederived the familiar Taylor series expansion $f(z) = \sum_{k=0}^{\infty} \dfrac{f^{(k)}(z_0)}{k!} (z - z_0)^k$.
  - In principle, we can use these formulas to recenter arbitrary series expansions of analytic functions. But in most cases when we are only given f as a series expansion, it is very difficult to evaluate the resulting expressions for the coefficients.
  - If we have an explicit formula for f (e.g., if f is a rational function), it is generally much easier simply to make a translation and then apply our earlier method for finding power series expansions of rational functions centered at $z = 0$.

- Example: Find the power series centered at $z = 1$ up to order 4 for $f(z) = \dfrac{2 + z}{2 - z}$.
  - Here, since we have an explicit formula for f, it is easier to evaluate a change of variables.
  - By setting $w = z - 1$, we equivalently want to find the power series expansion centered at $w = 0$ for $f(z) = f(w + 1) = \dfrac{3 + w}{1 - w}$.
  - This series is $(3 + w)(1 + w + w^2 + w^3 + w^4 + \cdots) = 3 + 4w + 4w^2 + 4w^3 + 4w^4 + \cdots$.
  - Rewriting in terms of z yields $f(z) = \boxed{3 + 4(z - 1) + 4(z - 1)^2 + 4(z - 1)^3 + 4(z - 1)^4 + \cdots}$.

2.3 Elementary Functions as Power Series

- Now that we have established the fundamental properties of power series as analytic functions, we can construct and study complex analogues of familiar real-valued elementary functions as power series.
  - In a fairly strong sense, there is an essentially unique way to extend a real-valued elementary function to one defined by a complex power series.
  - More precisely, if f is a real-valued function on the real line, there is at most one power series $\tilde{f} = \sum_{n=0}^{\infty} a_n z^n$ with positive radius of convergence that agrees with f on an open real interval $(-\epsilon, \epsilon)$ for some $\epsilon > 0$. (If there were two such series, then their difference would be zero on the entire interval $(-\epsilon, \epsilon)$ hence by our uniqueness result the difference would be identically zero.)
  - In particular, for any real function f defined by a Taylor series $\sum_{n=0}^{\infty} a_n x^n$ that converges on some open interval to the value of f, the "right" definition is to take its complex extension as $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Since this power series agrees with $f(x)$ on an open interval of the real line, this is the only possible choice of $f(z)$ by the observation above. Additionally, the series $f(z)$ has positive radius of convergence and is therefore holomorphic, so we retain the ability to manipulate it just as with the original real Taylor series.

- Moreover, if we apply our uniqueness observation to function identities that hold for real arguments, we see that those identities will necessarily carry over to complex arguments as well. (This phenomenon is sometimes referred to as the "principle of the persistence of functional relations".)

2.3.1 Complex Exponentials and Trigonometric Functions

- We start by defining the complex exponential as a power series, which (per our discussion above) is completely determined by the real Taylor series expansion of the exponential:
- Definition: For $z \in \mathbb{C}$, we define the complex exponential $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.
  - This series has radius of convergence ∞, as we have previously calculated. Therefore, we can differentiate termwise to obtain the expected $\frac{d}{dz}[e^z] = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$.
  - Furthermore, for any fixed $w \in \mathbb{C}$, the function $e^{z+w} - e^z e^w$ is analytic on all of $\mathbb{C}$ hence has a local power series expansion at 0. But this function is identically zero on the real line (per the usual properties of the real exponential function) hence by the uniqueness of power series expansions, it must be identically zero on all of $\mathbb{C}$. Thus, we have the usual identity $e^{z+w} = e^z e^w$ for any $z, w \in \mathbb{C}$.
  - Of course, it is possible (although very tedious) to verify this identity directly by expanding out the power series for e^{z+w}.
  - In particular we also recover all of the usual multiplicative properties of the complex exponential: for example, taking $z = -w$ and rearranging yields $e^{-z} = 1/e^z$.
  - We may also (re)derive Euler's identity: setting $z = i\theta$ produces

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots = \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right] = \cos\theta + i\sin\theta$$

    using the real Taylor series expansions for $\sin\theta$ and $\cos\theta$.
  - Then for x, y real we have $e^{x+iy} = e^x e^{iy} = e^x[\cos y + i\sin y]$, which establishes that our definition of the complex exponential as a power series agrees with that of our earlier approach.
  - As an immediate consequences we see that $\overline{e^z} = e^{\overline{z}}$ and that $|e^z| = e^{\mathrm{Re}(z)}$, so in particular e^z is never equal to zero. But by using the polar form we can see that every nonzero complex number $r[\cos\theta + i\sin\theta]$ is in the image of e^z: explicitly, $e^{\ln(r)+i\theta} = r[\cos\theta + i\sin\theta]$ where $\ln(r)$ represents the real natural logarithm of the positive real number r.
  - The complex exponential is periodic with period $2\pi i$, since $e^{2\pi i} = 1$ and thus $e^{z+2\pi i} = e^z$ for all z. Indeed, if $e^z = 1$ for $z = x + iy$ then $1 = |e^z| = e^x$ so $x = 0$, and then $1 = e^{iy} = \cos y + i\sin y$ yields $y = 2\pi k$ for some integer k.
  - So we see that $e^z = 1$ if and only if $z = 2\pi ik$ for some integer k, and so as a consequence we have $e^{z_1} = e^{z_2}$ if and only if $e^{z_2 - z_1} = 1$ if and only if $z_2 = z_1 + 2\pi ik$ for some integer k.
- Example: Find all $z \in \mathbb{C}$ with $e^z = 2 + 2i$.
  - Writing $2 + 2i$ in exponential form yields $2 + 2i = 2\sqrt{2}e^{i\pi/4} = e^{\ln(2\sqrt{2})+i\pi/4}$.
  - By the remarks above this is equivalent to $z - [\ln(2\sqrt{2}) + i\pi/4] = 2k\pi i$ for some integer k, which gives $z = \boxed{\ln(2\sqrt{2}) + (\pi/4 + 2k\pi)i}$.
- Example: Find all $z \in \mathbb{C}$ with $e^{(1+i)z} = 2$.
  - Since $2 = e^{\ln 2}$, the equation is equivalent to $e^{(1+i)z} = e^{\ln(2)}$ which in turn is equivalent to $(1+i)z - \ln(2) = 2k\pi i$ for some integer k.
  - This yields $z = \boxed{\frac{\ln(2) + 2k\pi i}{1+i} = \frac{1}{2}\ln(2)\cdot(1-i) + k\pi(1+i)}$ for some integer k.

- We can also extend the definitions of sine and cosine in the same way as the exponential:
- Definition: For $z \in \mathbb{C}$, we have the complex sine $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$ and cosine $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$.
  - From the series expansions above we can easily see that $e^{iz} = \cos(z) + i\sin(z)$ and $e^{-iz} = \cos(z) - i\sin(z)$, and so we have $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$ and $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$. Since e^{iz} and e^{-iz} have radius of convergence ∞, so do $\sin(z)$ and $\cos(z)$ (this can also be seen directly from the series expansions).
  - We also have the usual relations $\frac{d}{dz}[\sin(z)] = \cos(z)$ and $\frac{d}{dz}[\cos(z)] = -\sin(z)$ by differentiating termwise, or, more easily, via $\frac{d}{dz}[\sin(z)] = \frac{d}{dz}[\frac{e^{iz} - e^{-iz}}{2i}] = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos(z)$ and $\frac{d}{dz}[\cos(z)] = \frac{d}{dz}[\frac{e^{iz} + e^{-iz}}{2}] = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin(z)$.
  - All of the familiar trigonometric identities also hold as well: the Pythagorean identity $\sin^2(z) + \cos^2(z) = 1$, the negation identities $\sin(-z) = -\sin(z)$ and $\cos(-z) = \cos(z)$, and the addition formulas $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$ and $\sin(z+w) = \sin(z)\cos(w) - \cos(z)\sin(w)$.
  - Each of these identities follows by noting that the difference between the two sides (for fixed w in the case of the addition formulas) is an analytic function of z on all of $\mathbb{C}$ that is identically zero on the real line, hence by our uniqueness results is identically zero on all of $\mathbb{C}$.
  - One can also establish the identities directly using the series expansions for sine and cosine (which is extremely tedious), or by reducing them to properties of the complex exponential function (which is less so).
  - In particular, since $\sin(2\pi) = 0$ and $\cos(2\pi) = 1$, by the addition formulas we see that $\cos(z + 2\pi) = \cos(z)$ and $\sin(z + 2\pi) = \sin(z)$ for all $z \in \mathbb{C}$, so sine and cosine remain periodic with period 2π as functions on $\mathbb{C}$. Likewise, since $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$ we have the usual complement formulas $\cos(\pi/2 - z) = \sin(z)$ and $\sin(\pi/2 - z) = \cos(z)$.
  - Indeed, we can also see that the real zeroes of sine (namely $k\pi$ for integers k) are actually the only complex zeroes: $\sin(z) = 0$ if and only if $e^{iz} = e^{-iz}$ if and only if $iz = (-iz) + 2k\pi i$ for some integer k, and this last condition is equivalent to $z = k\pi$ for some integer k.
  - By the complement formulas this also means the real zeroes of cosine are its only complex zeroes as well.
- Example: Find all $z \in \mathbb{C}$ with $\cos(z) = 3$.
  - From the definition we have $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$ so setting $w = e^{iz}$ yields $3 = \frac{w + w^{-1}}{2}$ hence $w^2 + 6w + 1 = 0$ so that $w = -3 \pm \sqrt{8}$.
  - The solutions to $e^{iz} = -3 - \sqrt{8}$ are $z = (\pi + 2\pi k) + i\ln(3 + \sqrt{8})$ for $k \in \mathbb{Z}$ while the solutions to $e^z = -3 + \sqrt{8} = -1/(3 + \sqrt{8})$ are $z = (\pi + 2\pi k) - i\ln(3 + \sqrt{8})$ for $k \in \mathbb{Z}$.
  - So the full set of solutions is $z = \boxed{(\pi + 2\pi k) \pm i\ln(3 + \sqrt{8})}$ for $k \in \mathbb{Z}$.
- We can also define the other complex trigonometric functions in the usual way in terms of sine and cosine:
- Definition: For $z \in \mathbb{C}$ we define $\tan(z) = \frac{\sin(z)}{\cos(z)}$, $\sec(z) = \frac{1}{\cos(z)}$, $\cot(z) = \frac{\cos(z)}{\sin(z)}$, and $\csc(z) = \frac{1}{\sin(z)}$.
  - From our analysis above we see that all of these functions are analytic and holomorphic on their entire domains, which for $\tan(z)$ and $\sec(z)$ are all $z \neq \pi/2 + \pi k$ and for $\cot(z)$ and $\csc(z)$ are all $z \neq k\pi$ for integers k.
  - Furthermore, since all of the usual identities for the complex sine and cosine still hold, the same is true for all of the usual identities involving these functions, such as the derivatives $\frac{d}{dz}[\tan(z)] = \sec^2(z)$, $\frac{d}{dz}[\sec(z)] = \sec(z)\tan(z)$, $\frac{d}{dz}[\cot(z)] = -\csc^2(z)$, $\frac{d}{dz}[\csc(z)] = -\csc(z)\cot(z)$ and the Pythagorean relations $1 + \tan^2(z) = \sec^2(z)$ and $1 + \cot^2(z) = \csc^2(z)$.

- We will also remark that the identities $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$ and $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$ are reminiscent of the definitions of the hyperbolic trigonometric functions. Indeed, there is a quite natural relationship between them:

- Definition: For $z \in \mathbb{C}$ we have the hyperbolic sine $\sinh(z) = \dfrac{e^z - e^{-z}}{2}$ and cosine $\cosh(z) = \dfrac{e^z + e^{-z}}{2}$.
  - Note that these definitions are the same as the usual ones for a real variable. We can see easily that $\dfrac{d}{dz}[\sinh(z)] = \cosh(z)$ and $\dfrac{d}{dz}[\cosh(z)] = \sinh(z)$.
  - The real-valued functions are defined in this way because they yield a parametrization $x = \cosh(t)$, $y = \sinh(t)$ of the hyperbola $x^2 - y^2 = 1$ analogous to the parametrization $x = \cos(t)$, $y = \sin(t)$ of the circle $x^2 + y^2 = 1$.
  - The connection between the hyperbolic sine and cosine with the regular sine and cosine in the real case is purely by analogy, but in the complex case we see easily that $\cosh(iz) = \cos(z)$ and $\sinh(iz) = i\sin(z)$, or equivalently, $\sin(iz) = i\sinh(z)$ and $\cos(iz) = \cosh(z)$.
  - Using these formulas we can easily obtain the hyperbolic analogues of the usual trigonometric identities (which otherwise require quite a bit more algebra): for example, $\sin^2(z) + \cos^2(z) = 1$ immediately yields $-\sinh^2(z) + \cosh^2(z) = 1$, while $\sin(2z) = 2\sin(z)\cos(z)$ yields $\sinh(2z) = 2\sinh(z)\cosh(z)$.
  - Likewise we can define the other hyperbolic trigonometric functions in the usual way: $\tanh(z) = \dfrac{\sinh(z)}{\cosh(z)}$, $\operatorname{sech}(z) = \dfrac{1}{\cosh(z)}$, $\coth(z) = \dfrac{\cosh(z)}{\sinh(z)}$, and $\operatorname{csch}(z) = \dfrac{1}{\sinh(z)}$.

- Example: Find all $z \in \mathbb{C}$ with $\sinh(z) = 4/3$.
  - From the definition we have $\dfrac{4}{3} = \dfrac{e^z - e^{-z}}{2}$ so setting $w = e^z$ yields $\dfrac{4}{3} = \dfrac{w - w^{-1}}{2}$ hence $w^2 - \dfrac{8}{3}w - 1 = 0$ so that $w = 3, -1/3$.
  - The solutions to $e^z = 3$ are $z = \ln(3) + 2\pi ik$ for $k \in \mathbb{Z}$ while the solutions to $e^z = -1/3$ are $z = \ln(1/3) + (\pi + 2\pi k)i$ for $k \in \mathbb{Z}$. So the full set of solutions is $z = \boxed{\ln(3) + 2\pi ik,\ \ln(1/3) + (\pi + 2\pi k)i}$ for $k \in \mathbb{Z}$.

2.3.2 The Complex Logarithm, Complex Powers

- We have now constructed a wide slate of elementary functions, such as the complex exponential and trigonometric functions, in analogy to the familiar real-valued exponential and trigonometric functions. Although there are various different choices of definitions (e.g., as convergent power series, as solutions to differential equations, as limits), all of these definitions can be shown to be equivalent.
  - However, the real-valued natural logarithm has several different definitions, and not all of these yield feasible options for defining a complex logarithm function.
  - For example, the natural logarithm is often defined as $\ln x = \int_1^x \frac{dt}{t}$: i.e., as an antiderivative of the function $f(x) = \dfrac{1}{x}$ for $x \neq 0$. However, there is no power series, or even Laurent series, centered at $z = 0$ whose derivative is $f(z) = \dfrac{1}{z}$.
  - As we will show later, in fact $f(z) = \dfrac{1}{z}$ does not have any antiderivative that is defined on the domain of $f(z)$, namely $\mathbb{C}\backslash\{0\}$, or even on an open region containing any circle centered at the origin.

- The most natural choice is to define the complex logarithm to be the inverse of the complex exponential function e^z.
  - However, unlike the real exponential e^x, which is one-to-one on $\mathbb{R}$ and thus has a well defined inverse function $\ln x$, the complex exponential e^z is periodic on $\mathbb{C}$ and thus does not have an inverse function.

- Instead, we must settle with taking the complex logarithm to be a multivalued function.
- Fortunately, this is still a familiar situation, since all of the trigonometric functions are periodic (with period 2π or π, depending on the function), so their inverses are also naturally multivalued functions.

- Definition: The complex logarithm $\log(z)$ is defined to be the multivalued function that is the inverse of the complex exponential e^z. For any $z \in \mathbb{C}$, $\log(z)$ is defined to be the set of all $w \in \mathbb{C}$ with $e^w = z$.
  - Notation: We will denote the real-valued natural logarithm function (with domain the positive real numbers) as $\ln x$ to keep its notation different from the multivalued complex logarithm function $\log z$.
  - Since the image of e^z consists of all nonzero complex numbers, the domain of $\log(z)$ is $\mathbb{C}\backslash\{0\}$.
  - Also, because $e^{z_1} = e^{z_2}$ if and only if $z_2 = z_1 + 2\pi i k$ for some integer k, we see that the set of values in $\log(z)$ is of the form $w + 2\pi i k$ for $k \in \mathbb{Z}$ where w is some individual value with $e^w = z$.
  - Using this observation, we can then explicitly compute the complex logarithm by writing $z \neq 0$ in polar (or equivalently exponential) form: for $z = re^{i\theta}$ we have $z = e^{\ln r + i\theta}$, so one value in the set $\log(z)$ is $\ln r + i\theta$ hence by the remark above, the full set is $\log(z) = \{\ln r + i(\theta + 2k\pi) \, : \, k \in \mathbb{Z}\}$.
  - From the exponential identity $e^{z+w} = e^z e^w$ we immediately obtain the corresponding logarithm identity $\log(zw) = \log(z) + \log(w)$, where the identity is to be interpreted as an equality of sets, in the sense that the set of all possible values of $\log(z) + \log(w)$ yields the set of possible values of $\log(zw)$.
  - Also, we see $\mathrm{Re}(\log z) = \ln |z|$ for any nonzero z and any value of the complex logarithm $\log z$.

- Example: Find all complex values of $\log(-1)$.
  - In polar form we have $-1 = e^{i\pi}$, so by the above, we have $\log(-1) = \boxed{(\pi + 2k\pi)i}$ for $k \in \mathbb{Z}$.

- Example: Find all complex values of $\log(\sqrt{3} - i)$.
  - In polar form we have $\sqrt{3} - i = 2e^{11i\pi/6}$, so by the above, we have $\log(\sqrt{3} - i) = \boxed{\ln(2) + (11\pi/6 + 2k\pi)i}$ for $k \in \mathbb{Z}$.

- By using the complex logarithm we can give a general definition for complex powers.
  - The idea is that we would like the rule $\log(a^b) = b \log a$ to hold for arbitrary complex a and b, in analogy with the corresponding identity for real logarithms.

- Definition: If $a, b \in \mathbb{C}$ with $a \neq 0$, we define a^b to be $e^{b \log a}$.
  - We note that in general, the quantity a^b will have multiple possible values corresponding to the different possible values of $\log a$.

- Example: Find all possible complex values of $(-1)^i$.
  - We have $\log(-1) = (\pi + 2k\pi)i$ for $k \in \mathbb{Z}$ as calculated above. So per the definition, we have $(-1)^i = e^{i \cdot (\pi + 2k\pi)i} = e^{-(\pi + 2k\pi)}$ for $k \in \mathbb{Z}$.
  - Explicitly, the possible values are $\dots, e^{-3\pi}, e^{-\pi}, e^{\pi}, e^{3\pi}, \dots$. Note, interestingly, that $(-1)^i$ has infinitely many distinct *real* values!

- Example: Find all possible complex values of $1^{1/4}$.
  - Since $\log(1) = 2\pi k i$, we have $1^{1/4} = e^{2\pi k i/4}$ for $k \in \mathbb{Z}$.
  - Unlike above, there are only four possible values of this expression: $e^0, e^{i\pi/2}, e^{i\pi}, e^{3i\pi/2}$, which are simply $1, i, -1, -i$. Quite sensibly, these are the four complex fourth roots of unity.

- We can obtain single-valued complex logarithm functions by restricting the domain of the exponential function to a region on which it is one-to-one, or equivalently, by selecting a specific value of the complex logarithm $\log(z)$ as output for each particular z.

- Our primary motivation is so that we can establish the continuity and differentiability of these various "branches" of the complex logarithm.
- First, we observe that it is not possible to make a continuous selection of a logarithm function on any circle $|z| = r$ for a fixed $r > 0$: the real part of the logarithm must be $\ln r$ (per our discussion above), and the imaginary part at $z = re^{i\theta}$ must be among $\{\dots, \theta - 2\pi, \theta, \theta + 2\pi, \theta + 4\pi, \dots\}$.
- If the value at $z = r$ is c, then if we continuously increase θ from 0, then the imaginary part must increase continuously from c at the same rate: however, at $\theta = 2\pi$, we have returned to $z = r$ but the imaginary part is now $c + 2\pi \neq c$.
- Therefore, any selection of a logarithm function must introduce some discontinuities.

• A fairly natural choice, motivated by our formula above as $\log(z) = \{\ln r + i(\theta + 2k\pi) : k \in \mathbb{Z}\}$, would be simply to select the value of the logarithm whose imaginary part lies in $[0, 2\pi)$. This yields what is usually called the principal complex logarithm:

• Definition: For nonzero $z \in \mathbb{C}$, the principal complex logarithm, sometimes denoted $\mathrm{Log}(z)$, is the unique value $w \in \mathbb{C}$ such that $e^w = z$ and $0 \le \mathrm{Im}(w) < 2\pi$.

- In other words, $\mathrm{Log}(z)$ is the inverse function of the restriction of e^z to the vertical strip $0 \le \mathrm{Im}(z) < 2\pi$, on which it is one-to-one.
- Examples: We have $\mathrm{Log}(-1) = i\pi$ and $\mathrm{Log}(\sqrt{3} - i) = \ln(2) + 11i\pi/6$.
- We can see that the principal complex logarithm is discontinuous on the positive real axis, since if we approach $z = r$ from above (i.e., along a path with positive imaginary part) the value of $\mathrm{Log}(z)$ will approach $\ln r$, while if we approach $z = r$ from below (i.e., along a path with negative imaginary part) the value of $\mathrm{Log}(z)$ will approach $\ln r + 2\pi i$.
- This type of discontinuity is called a branch cut: along the path from 0 to ∞ that lies along the positive real axis, we have (essentially) "cut" the branch of the complex logarithm to ensure it is single-valued.
- By selecting different possible regions for the imaginary part of the logarithm we can arrange different branch cuts from 0 to ∞.
- For example, if we instead choose to require the imaginary part of the logarithm lie in $[-\pi/2, 3\pi/2)$, then the branch cut is along the negative imaginary axis. More generally, picking the imaginary part in $[\alpha, \alpha + 2\pi)$ will put the branch cut along the ray $\theta = \alpha$, where θ represents the usual polar angle.

• We claim that the principal complex logarithm (and indeed any other possible choice of complex-valued logarithm function) is continuous and holomorphic except along its branch cut.

- This claim follows immediately from the rule for differentiating inverse functions.
- Explicitly, suppose that $f(z)$ is one-to-one and holomorphic on a region R, so that f^{-1} exists. Suppose $f(z_0) = w_0$, that z_0 is an interior point of R, and that $f'(z_0) \neq 0$.
- Then we have $(f^{-1})'(w_0) = \lim_{w \to w_0} \dfrac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{z \to z_0} \dfrac{z - z_0}{f(z) - f(z_0)} = \dfrac{1}{f'(z_0)} = \dfrac{1}{f'(f^{-1}(w_0))}$ by making the substitution[4] $w = f(z)$.
- Applying this calculation where $f(z) = e^z$ restricted to an appropriate region R where it is one-to-one shows that for $g(z) = f^{-1}(z)$, we have $g'(z) = \dfrac{1}{f'(f^{-1}(z))} = \dfrac{1}{e^{\mathrm{Log}(z)}} = \dfrac{1}{z}$ for all z such that $f^{-1}(z)$ lies in the interior of R (which is equivalent to saying that z is not on the branch cut), as expected.
- In particular, the principal complex logarithm is holomorphic on $\mathbb{C}\backslash[0, \infty)$ with derivative $\mathrm{Log}'(z) = \dfrac{1}{z}$, and hence it is also continuous there.

Well, you're at the end of my handout. Hope it was helpful.

[4] One may object that this calculation presupposes the continuity of f^{-1} at w_0. However, continuity of f^{-1} follows from the fact that f is differentiable and one-to-one: this can be proven directly but it is rather painful to do; in our situation it is essentially the Inverse Function Theorem for $\mathbb{R}^2$.