Calculus III (part 4): Vector Calculus (by Evan Dummit, 2021, v. 4.01)

Contents

4 Vector Calculus

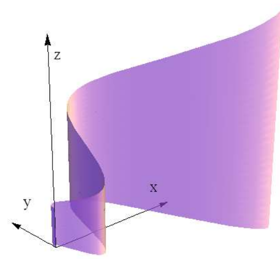
Our motivating problem for multivariable integration was to generalize the idea of integration to more complicated regions in space, or (more succinctly) to integrate a function over a region. We might also ask whether there is a simple way to integrate a function over an arbitrary curve in the plane or in space, and whether there is a way to integrate a function over an arbitrary surface in space. The answer (as it always has been to this point) is yes: the generalization of single-variable integration to arbitrary curves is called a line integral, and the generalization of double integration to arbitrary surfaces is called a surface integral.

After introducing line and surface integrals, we will then discuss vector fields (which are vector-valued functions in 2-space and 3-space) which provide a useful model for the flow of a fluid through space. The principal applications of line and surface integrals are to the calculation of the work done by a vector field on a particle traveling through space, the flux of a vector field across a curve or through a surface, and the circulation of a vector field along a curve.

Finally, we discuss several generalizations of the Fundamental Theorem of Calculus: the Fundamental Theorem of Calculus for line integrals, Green's Theorem, Gauss's Divergence Theorem, and Stokes's Theorem. Collectively, these theorems unify all of the different notions of integration, as they each relate the integral of a function on a region to the integral of an antiderivative of the function on the region's boundary.

4.1 Line Integrals

- The motivating problem for our discussion of line integrals is: given a parametric curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ and a function $f(x, y)$, if we "build a surface" along the curve with height given by the function $z = f(x, y)$, how can we calculate the area of this surface? (This is a natural generalization of our typical single-variable integration problem, in which we build the "surface" inside a plane, thus making it the area under a curve.)
  - Here is an example (for visualization), with $\mathbf{r}(t) = \left\langle t^2,\, t\cos(2\pi t)\right\rangle$, $f(x, y) = t^2 + 1$, for $0 \le t \le \dfrac{3}{2}$:



  - Another closely related question is: given a parametric curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ and a function $f(x, y, z)$, how can we calculate the average value of $f(x, y, z)$ on the curve?
  - A third question: given a thin wire shaped along some curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ with variable density $\delta(x, y)$, what is the wire's mass, and what are its moments about the coordinate axes?

- As with all other types of integrals we have examined so far, we use Riemann sums to give the formal definition of the line integral of a function $f(x, y)$ on a plane curve C. (Also as before, we will use the formal definition as infrequently as possible!)
  - The idea is to approximate the curve with straight line segments, sum (over all the segments) the function value times the length of the segment, and then take the limit as the segment lengths approach zero.
  - Definition: For a curve C, a partition of C into n pieces is a list of points (x_0, y_0), ... , (x_n, y_n) on C, with the nth segment having length $\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$. The norm of the partition P is the largest number among all of the segment lengths in P.
  - Definition: For $f(x, y)$ a continuous function and P a partition of the curve C, we define the Riemann sum of $f(x, y)$ on D corresponding to P to be $\mathrm{RS}_P(f) = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta s_k$.
  - Definition: For a function $f(x, y)$, we define the line integral of f on the curve C, denoted $\displaystyle\int_C f(x, y)\, ds$, to be the value of L such that, for every $\epsilon > 0$, there exists a $\delta > 0$ (depending on ϵ) such that for every partition P with $\text{norm}(P) < \delta$, we have $|RS_P(f) - L| < \epsilon$.
  - Remark: It can be proven (with significant effort) that, if $f(x, y)$ is continuous and the curve C is smooth, then a value of L satisfying the hypotheses actually does exist.
  - Remark: The differential ds in the definition of the line integral is the "differential of arclength", which we discussed earlier in our study of vector-valued functions.

- In exactly the same way, we can use Riemann sums to give a formal definition of the line integral along a curve C in 3-space. (We simply add the appropriate z-terms to all the definitions.)

- Like with the other types of integrals, line integrals have a number of formal properties which can be deduced from the Riemann sum definition. Specifically, for D an arbitrary constant and $f(x, y)$ and $g(x, y)$ continuous functions, the following properties hold:
  - Integral of constant: $\int_C D\, ds = D \cdot \text{Arclength}(C)$.
  - Constant multiple of a function: $\int_C D\, f(x, y)\, ds = D \cdot \int_C f(x, y)\, ds$.
  - Addition of functions: $\int_C f(x, y)\, ds + \int_C g(x, y)\, ds = \int_C [f(x, y) + g(x, y)]\, ds$.

- Subtraction of functions: $\int_C f(x,y)\,ds - \int_C g(x,y)\,ds = \int_C [f(x,y) - g(x,y)]\,ds$.
- Nonnegativity: if $f(x,y) \geq 0$, then $\int_C f(x,y)\,ds \geq 0$.
- Union: If C_1 and C_2 are curves such that C_2 starts where C_1 ends, and C is the curve obtained by gluing the curves end-to-end, then $\int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds = \int_C f(x,y)\,ds$.
- Remark: These same properties also all hold for line integrals of a function $f(x,y,z)$ in 3-space.

- The key observation is that we can reduce calculations of line integrals to "traditional" single integrals:

- Proposition (Line Integrals in the Plane): If the curve C can be parametrized as $x = x(t)$, $y = y(t)$ for $a \leq t \leq b$, then $\displaystyle\int_C f(x,y)\,ds = \int_a^b f(x(t),\, y(t))\,\frac{ds}{dt}\,dt$, where $\dfrac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$ is the derivative of arclength.

- Proposition (Line Integrals in 3-Space): If the curve C can be parametrized as $x = x(t)$, $y = y(t)$, $z = z(t)$ for $a \leq t \leq b$, then $\displaystyle\int_C f(x,y,z)\,ds = \int_a^b f(x(t),\, y(t),\, z(t))\,\frac{ds}{dt}\,dt$, where $\dfrac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$ is the derivative of arclength.

    - The proof of both of these results is simply to observe that the Riemann sum $\displaystyle\sum_{k=1}^{n} f(x_k, y_k)\,\Delta s_k$ for the line integral $\int_C f(x,y)\,ds$ is also a Riemann sum $\displaystyle\sum_{k=1}^{n} f(x_k, y_k)\,\frac{\Delta s_k}{\Delta t_k}\Delta t_k$ for the integral $\int_a^b f(x(t),\, y(t))\,\dfrac{ds}{dt}\,dt$.
    - Equivalently: we have made a substitution in the integral by changing from s-coordinates to t-coordinates, where the differential changes using the rule $ds = \dfrac{ds}{dt}dt$.

- Thus, to evaluate the line integral of f on the curve C (i.e., the line integral $\int_C f(x,y,z)\,ds$), follow these steps:

    1. Parametrize the curve C as a function of t, as $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$ for $a \leq t \leq b$.
    2. Write the function f in terms of t: $f(x,y,z) = f(x(t), y(t), z(t))$.
    3. Write the differential $ds = \dfrac{ds}{dt}\,dt = ||\mathbf{v}(t)||\,dt = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$ in terms of t.
    4. Evaluate the resulting single-variable integral $\int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt$.

- Example: Integrate the function $f(x,y,z) = yz - 6x$ along the curve $\mathbf{r}(t) = \langle t^3, 6t, 3t^2\rangle$ from $t = 0$ to $t = 1$.

    - We have $f(x,y,z) = yz - 6x = (6t)(3t^2) - 6t^3 = 12t^3$, and we also have $ds = \sqrt{(3t^2)^2 + (6)^2 + (6t)^2} = \sqrt{9t^4 + 36t^2 + 36} = 3t^2 + 6$.
    - The integral is therefore $\int_0^1 (12t^3)(3t^2 + 6)dt = \int_0^1 (36t^5 + 72t^3)\,dt = \boxed{24}$.

- Example: Integrate the function $f(x,y) = x^2 + y$ along the top half of the unit circle $x^2 + y^2 = 1$, starting at $(1,0)$ and ending at $(-1,0)$.

    - The unit circle is parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$: the range we want is $0 \leq t \leq \pi$.
    - We have $f(x,y) = x^2 + y = \cos^2 t + \sin t$, and we also have $ds = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$.
    - The integral is therefore $\int_0^\pi \left[\cos^2 t + \sin t\right] dt = \int_0^\pi \left[\dfrac{1 + \cos 2t}{2} + \sin t\right] dt = \boxed{\dfrac{\pi}{2} + 2}$.

- To find the average value of a function on a curve, we simply integrate the function over the curve, and then divide by the curve's arclength.

- Example: Find the average value of the function $f(x,y,z) = x^2 + y^2 + z^2$ along the line segment from $(1,-1,0)$ to $(2,2,1)$.

    - The direction vector for the line is $\mathbf{v} = \langle 2,2,1\rangle - \langle 1,-1,0\rangle = \langle 1,3,1\rangle$. Thus, we can parametrize the line segment as $\langle x,y,z\rangle = \langle 1,-1,0\rangle + t\,\langle 1,3,1\rangle$ for $0 \leq t \leq 1$.

- So the line segment is parametrized explicitly by $x = 1 + t$, $y = -1 + 3t$, $z = t$ for $0 \le t \le 1$.
- Now we set up the integral: the function is $f(x, y, z) = x^2 + y^2 + z^2 = (1 + t)^2 + (-1 + 3t)^2 + (t)^2 = 11t^2 - 4t + 2$.
- Since $x'(t) = 1$, $y'(t) = 3$, and $z'(t) = 1$, we also have $\frac{ds}{dt} = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$.
- The integral of f is therefore $\int_0^1 \left[11t^2 - 4t + 2\right] \sqrt{11} dt = \sqrt{11} \left[\frac{11}{3} t^3 - 2t^2 + 2t\right]\Big|_{t=0}^{1} = \frac{11\sqrt{11}}{3}$.
- To compute the average value, we divide by the arclength, which is $\int_0^1 1\, ds = \int_0^1 \sqrt{11} dt = \sqrt{11}$.
- Thus, the average value is $\boxed{\frac{11}{3}}$.

- We also have formulas for the mass and moments of a wire of variable density:
- Center of Mass and Moment Formulas (Thin Wire): Given a 1-dimensional wire of variable density $\delta(x, y, z)$ along a parametric curve C in 3-space:
  - The total mass M is given by $M = \int_C \delta(x, y, z)\, ds$.
  - The x-moment M_{yz} is given by $M_{yz} = \int_C x\, \delta(x, y, z)\, ds$.
  - The y-moment M_{xz} is given by $M_{xz} = \int_C y\, \delta(x, y, z)\, ds$.
  - The z-moment M_{xy} is given by $M_{xy} = \int_C z\, \delta(x, y, z)\, ds$.
  - The center of mass $(\bar{x}, \bar{y}, \bar{z})$ has coordinates $\left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$.
  - Note: For a wire in 2-space, the formulas are essentially the same (except without the z-coordinate), though the x-moment is denoted M_y and the y-moment is denoted M_x.

- Example: Find the total mass, and the center of mass, of a thin wire in the xy-plane having the shape of the unit circle with variable density $\delta(x, y) = 2 + x$.
  - We can parametrize the unit circle with $x = \cos t$, $y = \sin t$, so $\frac{ds}{dt} = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$.
  - The total mass M is $M = \int_C \delta(x, y)\, ds = \int_0^{2\pi} (2 + \cos t)\, dt = \boxed{2\pi}$.
  - The x-moment M_y is $M_y = \int_C x\, \delta(x, y)\, ds = \int_0^{2\pi} \cos t(2 + \cos t)\, dt = \left[2 \sin t + \frac{1}{2} t + \frac{1}{4} \sin(2t)\right]\Big|_{t=0}^{2\pi} = \pi$.
  - The y-moment M_x is $M_x = \int_C y\, \delta(x, y)\, ds = \int_0^{2\pi} \sin t(2 + \cos t)\, dt = \left[-2 \cos t - \frac{1}{4} \cos(2t)\right]\Big|_{t=0}^{2\pi} = 0$.
  - Therefore, the center of mass is $\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{1}{2}, 0\right)}$.

- We will also be interested in computing line integrals involving the differentials dx, dy, and dz rather than ds: namely, expressions of the form $\displaystyle\int_C f\, dx + g\, dy + h\, dz$.
- We evaluate such line integrals by making the appropriate substitutions: if C is parametrized by $x = x(t)$, $y = y(t)$, $z = z(t)$ for $a \le t \le b$, then the line integral $\displaystyle\int_C f\, dx + g\, dy + h\, dz$ is given by the single-variable integral $\displaystyle\int_a^b \left[f \frac{dx}{dt} + g \frac{dy}{dt} + h \frac{dz}{dt}\right] dt$.
- Example: Find $\int_C y\, dx + z\, dy + x^2\, dz$, where C is the curve $(x, y, z) = (t, t^2, t^3)$ ranging from $t = 0$ to $t = 1$.
  - We have $x = t$, $y = t^2$, and $z = t^3$, so that $dx = dt$, $dy = 2t\, dt$, and $dz = 3t^2\, dt$.
  - The integral is $\int_0^1 \left[t^2 \cdot dt + 3t^2 \cdot 2t\, dt + t^2 \cdot 3t^2\, dt\right] = \int_0^1 \left[t^2 + 6t^3 + 3t^4\right] dt = \boxed{\frac{73}{30}}$.

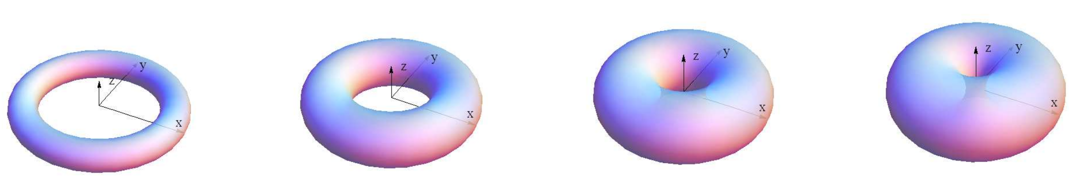
- Example: Find $\int_C x\,dy - y\,dx$, where C is the upper half of the ellipse $x^2/9 + y^2/16 = 1$, starting at $(3, 0)$ and ending at $(-3, 0)$.
  - This ellipse is parametrized by $\mathbf{r}(t) = \langle 3\cos t,\, 4\sin t\rangle$: the range we want is $0 \le t \le \pi$.
  - We have $x = 3\cos t$ and $y = 4\sin t$, so that $dx = -3\sin t\,dt$ and $dy = 4\cos t\,dt$.
  - The desired integral is $\int_0^\pi [3\cos t \cdot (4\cos t\,dt) - 4\sin t \cdot (-3\sin t\,dt)] = \int_0^\pi \left[12\cos^2 t + 12\sin^2 t\right] dt = \boxed{12\pi}$.

4.2 Surfaces and Surface Integrals

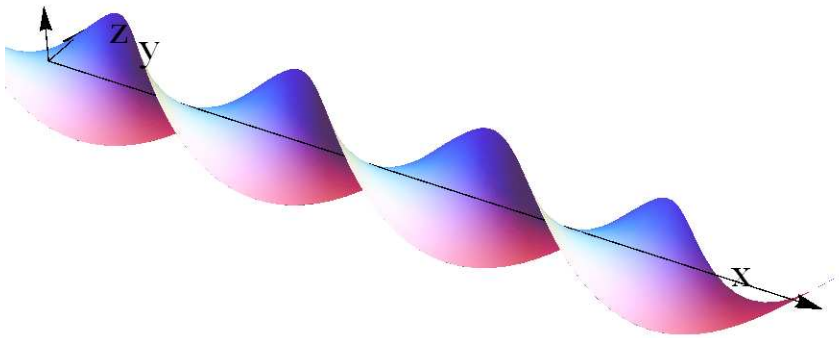
- We would now like to consider the problem of computing the integral of a function on a surface in 3-dimensional space. In a similar way to how we computed line integrals using (single) integrals, we will be able to compute surface integrals as double integrals.
- There are essentially two ways to describe a surface in 3-space: either as an implicit surface of the form $f(x, y, z) = c$, or as a parametric surface $\mathbf{r}(s, t) = \langle x(s, t),\, y(s, t),\, z(s, t)\rangle$ for two parameters s and t.
  - Note that the "explicit surface" $z = g(x, y)$ is simply a special case of the general implicit surface, since $g(x, y) - z = 0$ has the form $f(x, y, z) = c$ with $f(x, y, z) = g(x, y) - z$ and $c = 0$.
  - In cases where the functions x, y, and z are sufficiently simple or nice, it can be possible to eliminate the variables s and t from the system $x = x(s, t)$, $y = y(s, t)$, $z = z(s, t)$, and obtain an equation for the surface as an implicit surface $f(x, y, z) = c$.
  - We will also remark that parametric descriptions of surfaces are often easier to work with than implicit descriptions. For example, graphing a parametric surface requires only plugging in values for (s, t) and plotting the resulting points (x, y, z), whereas graphing an implicit surface requires finding solutions to the implicit equation, which is typically much harder.
- We will describe how to find parametrizations of some common surfaces, give the definition of a surface integral, and then show how to compute surface integrals on both parametric and implicit surfaces.

4.2.1 Parametric Surfaces

- If we graph a vector-valued function of two variables $\mathbf{r}(s, t) = \langle x(s, t),\, y(s, t),\, z(s, t)\rangle$ as s and t vary, we will obtain a surface in space (barring something strange happening).
- Example: The surface $\mathbf{r}(s, t) = \langle x_0, y_0, z_0\rangle + t\,\langle v_1, v_2, v_3\rangle + s\,\langle w_1, w_2, w_3\rangle$ is the plane passing through the point (x_0, y_0, z_0) that contains the two vectors $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$, provided that $\mathbf{v}$ and $\mathbf{w}$ are not parallel.
  - We could also describe the plane as an implicit surface of the form $ax + by + cz = d$, where $\langle a, b, c\rangle = \mathbf{v}\times\mathbf{w}$ is the normal vector to the plane and $d = ax_0 + by_0 + cz_0$.
  - There are many ways to describe a given plane as a parametric surface. For example, both of the parametrizations $\mathbf{r}(s, t) = \langle s,\, t,\, 1 - s - t\rangle$ and $\mathbf{r}(s, t) = \langle -3 + s - 2t,\, 2 + t + 2s,\, 2 + t - 3s\rangle$ describe the same plane $x + y + z = 1$.
- Example: For two positive "radius parameters" r and R with $r < R$, the surface defined parametrically by $\mathbf{r}(s, t) = \langle \cos(t)[R + r\cos(s)],\, \sin(t)[R + r\cos(s)],\, r\sin(s)\rangle$, for $0 \le t \le 2\pi$ and $0 \le s \le 2\pi$ is a donut-shaped surface called a torus.
  - It is the surface obtained by taking a vertical circle of radius r and moving its center along the circle $x^2 + y^2 = R^2$ in the xy-plane.
  - Four tori, with respective parameters (r, R) equal to $(1, 5)$, $(2, 5)$, $(3, 5)$, and $(4, 5)$, are plotted below:



* Example: The surface defined parametrically by $\mathbf{r}(s,t) = \langle \cos(s) + \cos(t),\ s+t,\ \sin(s)+\sin(t)\rangle$, for $0 \le t \le 4\pi$ and $0 \le s \le 4\pi$ is a helical ribbon:



* In general, it can be a somewhat involved problem to convert a geometric or verbal description of a surface into a parametrization: it is really more of an art form than a general procedure.
  * To parametrize parts of cylinders, cones, and spheres, it is almost always a very good idea to consider whether cylindrical or spherical coordinates can be of assistance.
  * Using translations and rescalings, we can also parametrize surfaces like ellipsoids.
* There are many different ways to parametrize the same surface, and which description is best will depend on what the parametrization will be used for.
  * For example, $x = s$, $y = t$, $z = \sqrt{s^2+t^2}$ parametrizes the cone $z = \sqrt{x^2+y^2}$, but so does the parametrization $x = s\cos t$, $y = s\sin t$, $z = s$.
  * If we want to describe the points lying over a rectangular region in the xy-plane, the first parametrization is more useful, but if we want to describe the points on the cone up to a specific height in the z-direction, the second parametrization is more useful.
* Example: Parametrize the portion of the cylinder $x^2+y^2=4$ lying between the planes $z=-2$ and $z=2$.
  * In cylindrical coordinates, we know that $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$.
  * Since the given cylinder has equation $r = 2$ in cylindrical coordinates, we see that a parametrization of the full cylinder is $x = 2\cos t$, $y = 2\sin t$, $z = s$, where $0 \le t \le 2\pi$ but with no restrictions on s. (Here we think of t as θ and s as z.)
  * To obtain just the portion with $-2 \le z \le 2$ we just restrict the range for s.
  * Thus the parametrization of the desired portion of the cylinder is $x = 2\cos t$, $y = 2\sin t$, $z = s$, where $0 \le t \le 2\pi$ and $-2 \le s \le 2$.
* Example: Parametrize the portion of the cylinder $x^2+y^2=4$ lying between the planes $z=y-2$ and $z=x+4$.
  * Like in the previous example, we take the parametrization of the full cylinder as $x = 2\cos t$, $y = 2\sin t$, $z = s$, and then restrict the ranges for s and t appropriately. In this case, we want the portion of the surface where $y-2 \le z \le x+4$.
  * It is straightforward to check that the two planes do not intersect inside the cylinder (since $y-2 \le 0$ inside the cylinder, while $x+4 \ge 2$).
  * So in this case, we take $0 \le t \le 2\pi$ and $2\sin t \le s \le 2\cos t + 4$.
* Example: Parametrize the sphere $x^2+y^2+z^2=9$.
  * In spherical coordinates, we know that $x = \rho\,\cos(\theta)\sin(\varphi)$, $y = \rho\,\sin(\theta)\,\sin(\varphi)$, $z = \rho\,\cos(\varphi)$.

- The sphere has equation $\rho = 3$, so we can immediately see that $x = 3\cos(t)\sin(s)$, $y = 3\sin(t)\sin(s)$, $z = 3\cos(s)$, with $0 \le t \le 2\pi$ and $0 \le s \le \pi$, will parametrize the sphere. (Here, we are thinking of t as θ and s as φ.)

- Example: Parametrize the sphere $(x-2)^2 + (y+1)^2 + (z-6)^2 = 4$.
  - It is not so easy to describe this sphere using spherical coordinates directly. However, if we shift the coordinates to center the sphere at the origin, we can easily write down the parametrization.
  - By translating back, we can see that $x = 2 + 2\cos(t)\sin(s)$, $y = -1 + 2\sin(t)\sin(s)$, $z = 6 + 2\cos(s)$, with $0 \le t \le 2\pi$ and $0 \le s \le \pi$, will parametrize the sphere.

- Example: Parametrize the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$.
  - It is again not so easy to write down the parametrization using any of our coordinate systems directly. However, if we rescale the coordinates by setting $x' = x/2$, $y' = y/3$, and $z' = z/4$, then we see $(x')^2 + (y')^2 + (z')^2 = 1$, and we can use spherical coordinates to parametrize the coordinates x', y', z'.
  - By rescaling back, we can see that $x = 2\cos(t)\sin(s)$, $y = 3\sin(t)\sin(s)$, $z = 4\cos(s)$, with $0 \le t \le 2\pi$ and $0 \le s \le \pi$, will parametrize this ellipsoid.

- Example: Parametrize the portion of the cone $z = 3\sqrt{x^2+y^2}$ that lies below the plane $z = 1 + x + y$.
  - In cylindrical, the equations are $z = 3r$ and $z = 2 + r\cos\theta + r\sin\theta$. They are equal when $3r = 2 + r\cos\theta + r\sin\theta$, or $r = \dfrac{2}{3 - \cos\theta - \sin\theta}$. (Note that $\sin\theta + \cos\theta \le \sqrt{2}$, so the denominator is never zero.)
  - The full surface is parametrized by $x = s\cos(t)$, $y = s\sin(t)$, $z = 3s$.
  - The portion under the plane corresponds to $0 \le s \le \dfrac{2}{3 - \cos t - \sin t}$, with $0 \le t \le 2\pi$.

- If we have a parametrization of a surface, we can use the parametrization to find the tangent plane to the surface at a given point.
  - The key observation is that if the surface S is parametrized by the vector-valued function $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$, then the two partial derivatives $\mathbf{r}_s = \dfrac{\partial \mathbf{r}}{\partial s}$ and $\mathbf{r}_t = \dfrac{\partial \mathbf{r}}{\partial t}$ are both tangent to the surface.
  - Therefore, the cross product $\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}$ will be perpendicular to the tangent plane, and is thus a normal vector for the tangent plane.

- Example: Find an equation for the tangent plane to the surface $\mathbf{r}(s,t) = \left\langle s\cos(t), s\sin(t), s^2\right\rangle$ when $s = 1$ and $t = \pi/2$.
  - We compute $\mathbf{r}_s(s,t) = \langle \cos t, \sin t, 2s\rangle$ and $\mathbf{r}_t(s,t) = \langle -s\sin t, s\cos t, 0\rangle$.
  - Thus, we see $\mathbf{r}_s(1,\pi/2) = \langle 0,1,2\rangle$, and $\mathbf{r}_t(1,\pi/2) = \langle -1,0,0\rangle$, and so the normal vector to the tangent plane is $\mathbf{n} = \langle 0,1,2\rangle \times \langle -1,0,0\rangle = \langle 0,-2,1\rangle$.
  - The tangent plane passes through the point on the surface where $s = 1$ and $t = \pi/2$, which is $\mathbf{r}(1,\pi/2) = \langle 0,1,1\rangle$.
  - Thus, an equation for the tangent plane is given by $\boxed{0(x-0) - 2(y-1) + 1(z-1) = 0}$ or equivalently $\boxed{-2y + z = -1}$.

- Example: Find an equation for the plane tangent to the surface $\mathbf{r}(s,t) = \left\langle s^2, 2st, t^3\right\rangle$ at the point $(4,4,-1)$.
  - First, we need to find the values of s and t at the point $(4,4,-1)$. If $\langle 4,4,-1\rangle = \left\langle s^2, 2st, t^3\right\rangle$ then we see $t^3 = -1$ so $t = -1$, and then $2st = 4$ gives $s = -2$.

- Now, we have $\mathbf{r}_s(s,t) = \langle 2s, 2t, 0\rangle$ and $\mathbf{r}_t(s,t) = \langle 0, 2s, 3t^2\rangle$, so $\mathbf{r}_s(-2,-1) = \langle -4,-2,0\rangle$ and $\mathbf{r}_t(-2,-1) = \langle 0,-4,3\rangle$.
- Thus, the normal vector to the tangent plane is $\mathbf{n} = \langle -4,-2,0\rangle \times \langle 0,-4,3\rangle = \langle -6,12,16\rangle$.
- Thus, an equation for the tangent plane is given by $\boxed{-6(x-4)+12(y-4)+16(z+1)=0}$ or equivalently $\boxed{-6x+12y+16z=8}$.

4.2.2 Surface Integrals

- The motivating problem for our discussion of surface integrals is as follows: given a parametric surface $\mathbf{r}(s,t) = \langle x(s,t),\, y(s,t),\, z(s,t)\rangle$ and a function $f(x,y,z)$, we would like to integrate the function on that surface. Like with line integrals, we have two natural applications: computing the average value of a function on the surface, and analyzing the physical properties of a thin surface with variable density.
  - As with all the other types of integrals, the idea is to approximate the surface with small "patches", sum (over all the patches) the function value times the area of the patch, and then take the limit as the patch sizes approach zero.
  - Definition: For a parametric surface S, a partition of S into n pieces is a list of disjoint subregions inside S, where the kth subregion corresponds to $s_k \le s \le s_k'$, $t_k \le t \le t_k'$, and has area $\Delta\sigma_k$. The norm of the partition P is the largest number among the areas of the rectangles in P.
  - Definition: For $f(x,y,z)$ a continuous function and P a partition of the surface S, we define the Riemann sum of $f(x,y,z)$ on R corresponding to P to be $\mathrm{RS}_P(f) = \sum_{k=1}^{n} f(\mathbf{r}(s_k,t_k))\,\Delta\sigma_k$.
  - Definition: For a function $f(x,y,z)$, we define the surface integral of f on S, denoted $\iint_S f\,d\sigma$, to be the value of L such that, for every $\epsilon > 0$, there exists a $\delta > 0$ (depending on ϵ) such that for every partition P with $\mathrm{norm}(P) < \delta$, we have $|RS_P(f) - L| < \epsilon$.
  - Remark: It can be proven (with significant effort) that, if $f(x,y,z)$ is continuous, then a value of L satisfying the hypotheses actually does exist.
- As with all of the other types of integrals, surface integrals possess some formal properties. For any continuous f and g and any constant C, we have the following:
  - Integral of constant: $\iint_S C\,d\sigma = C \cdot \mathrm{Area}(S)$.
  - Constant multiple of a function: $\iint_S C\,f\,d\sigma = C \cdot \iint_S f\,d\sigma$.
  - Addition of functions: $\iint_S f\,d\sigma + \iint_S g\,d\sigma = \iint_S [f+g]\,d\sigma$.
  - Subtraction of functions: $\iint_S f\,d\sigma - \iint_S g\,d\sigma = \iint_S [f-g]\,d\sigma$.
  - Nonnegativity: if $f \ge 0$, then $\iint_S f\,d\sigma \ge 0$.
  - Union: If S_1 and S_2 don't overlap and have union S, then $\iint_{S_1} f\,d\sigma + \iint_{S_2} f\,d\sigma = \iint_S f\,d\sigma$.
- We were able to reduce line integral calculations to standard one-variable integrals. We can similarly reduce calculations of surface integrals to double integrals:
- Proposition (Parametric Surface Integrals): If $f(x,y,z)$ is continuous on the surface S which is parametrized as $\mathbf{r}(s,t) = \langle x(s,t),\, y(s,t),\, z(s,t)\rangle$, where S is described by a region R in st-coordinates, then the surface integral of f on S is

$$\iint_S f\,d\sigma \quad = \quad \iint_R f(x(s,t),\, y(s,t),\, z(s,t)) \left|\left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right|\right| \,dt\,ds.$$

  - The key step is to recognize the Riemann sum for the surface integral as the Riemann sum for a particular double integral.

- Ultimately, the differential of surface area $d\sigma = \left|\left|\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right|\right| dt\,ds$ arises from computing the area of a small patch in st-coordinates: when s changes slightly, the change in $\mathbf{r}$ is given by $\dfrac{\partial \mathbf{r}}{\partial s}$, and when t changes slightly, the change in $\mathbf{r}$ is given by $\dfrac{\partial \mathbf{r}}{\partial t}$.
- These two vectors form a small parallelogram that closely approximates the surface S, so the differential of surface area $d\sigma$ is roughly equal to the area of this parallelogram, which is $\left|\left|\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right|\right|$, times the differential $dt\,ds$.

- We can also calculate surface integrals over implicit surfaces of the form $g(x, y, z) = c$:

- Proposition (Implicit Surface Integrals): If $f(x, y, z)$ is continuous on the implicit surface S defined by $g(x, y, z) = c$, R is the projection of S into the xy-plane, and $\partial g/\partial z \neq 0$ on R, then the surface integral of f on S is

$$\iint_S f\,d\sigma \quad = \quad \iint_R f(x, y, z)\,\frac{||\nabla g||}{|\nabla g \cdot \mathbf{k}|}\,dy\,dx$$

where ∇g is the gradient of g and $\mathbf{k} = \langle 0, 0, 1\rangle$. (Thus, $\nabla g \cdot \mathbf{k} = \partial g/\partial z$.)

- The statement that $\partial g/\partial z \neq 0$ on R is equivalent to saying that the tangent plane to $g(x, y, z) = c$ is never vertical above R. In particular this implies that the surface never "doubles back" on itself over the region R.
- Thus for example, we could not use the method directly to compute a surface integral on the entire unit sphere, because it has a vertical tangent plane above its projection $x^2 + y^2 \leq 1$ in the xy-plane.
- This formula can be derived from the parametric surface integral formula: after some simplification, it is what one obtains by using the parametrization $\mathbf{r}(s, t) = \langle s, t, z(s, t)\rangle$, where $z(s, t)$ is defined implicitly via the relation $f(s, t, z(s, t)) = c$.

- Using these two results, we can reduce calculations of surface integrals to "traditional" double integrals: given a description of the surface S, we can convert it to a double integral using one of two methods:
  - For a parametric surface given in the form $\mathbf{r}(s, t) = \langle x(s, t),\ y(s, t),\ z(s, t)\rangle$:
    - Step 1: Find the bounds on s and t that parametrize the desired portion of the surface.
    - Step 2: Express the function $f(x, y, z)$ to be integrated in terms of (s, t).
    - Step 3: Find the differential of surface area $d\sigma = \left|\left|\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right|\right| ds\,dt$.
    - Step 4: Write down the integral $\iint_S f(x(s, t), y(s, t), z(s, t)) \left|\left|\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right|\right| ds\,dt$ and evaluate.
  - For an implicit surface given in the form $g(x, y, z) = c$:
    - Step 1: Sketch the surface, determine the shape of its projection R into the xy-plane, and make sure that the surface does not cover any part of the projection more than once.
    - Step 2: Evaluate the integral $\iint_R f(x, y, z)\,\dfrac{||\nabla g||}{|\nabla g \cdot \mathbf{k}|}\,dy\,dx$, where ∇g is the gradient of g and $\mathbf{k} = \langle 0, 0, 1\rangle$.
    - Note that the only variables allowed in the integral are x and y, so if the integrand has any z terms we must use the implicit equation $g(x, y, z) = c$ to get rid of them.
  - Note that, by swapping z with x or with y, the implicit surface procedure can also be used with a projection into the xz-plane or the yz-plane.
  - Also note that for a surface of the form $z = f(x, y)$, we could use either method.

- Example: Integrate the function $g(x, y, z) = z$ over the surface with parametrization $\mathbf{r}(s, t) = \langle \sin(t),\ \cos(t),\ s + t\rangle$ for $0 \leq t \leq 2\pi$ and $0 \leq s \leq \pi$.

- We have an explicit parametrization of the surface, so we use the parametric formula.
- On the surface, we have $z = s + t$ so $g(x, y, z) = z = s + t$.
- We have $\dfrac{\partial \mathbf{r}}{\partial s} = \langle 0, 0, 1\rangle$ and $\dfrac{\partial \mathbf{r}}{\partial t} = \langle \cos(t), -\sin(t), 1\rangle$, so $\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ \cos(t) & -\sin(t) & 1 \end{vmatrix} = \langle \sin(t), \cos(t), 0\rangle$. Then $\left|\left|\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right|\right| = 1$.
- The integral is therefore given by

$$\int_0^{2\pi}\int_0^{\pi} (s+t)\,ds\,dt = \int_0^{2\pi} \left[\frac{s^2}{2} + st\right]\Big|_{s=0}^{\pi} dt = \int_0^{2\pi} \left[\frac{\pi^2}{2} + \pi t\right] dt = \left[\frac{\pi^2}{2}t + \frac{\pi}{2}t^2\right]\Big|_{t=0}^{2\pi} = \boxed{3\pi^3}.$$

- Example: Integrate the function $f(x, y, z) = 8xy$ over the portion of the plane $2x + y + 2z = 1$ with $0 \le x \le 1$, $0 \le y \le 1$.
  - We use the implicit surface formula, with $g(x, y, z) = 2x + y + 2z - 1$.
  - We have $\nabla g = \langle 2, 1, 2\rangle$ so $||\nabla g|| = \sqrt{2^2 + 1^2 + 2^2} = 3$ and $|\nabla g \cdot \mathbf{k}| = 2$.
  - The desired integral is therefore $\int_0^1 \int_0^1 8xy \cdot (3/2)\,dy\,dx = \int_0^1 6x\,dx = \boxed{3}$.
- Example: Integrate the function $f(x, y, z) = xz$ over the portion of the plane $4x + 2y + z = 1$ with $0 \le x \le 1$, $0 \le y \le 1$.
  - We use the implicit surface formula, with $g(x, y, z) = 4x + 2y + z - 1$.
  - We have $\nabla g = \langle 4, 2, 1\rangle$ so $||\nabla g|| = \sqrt{4^2 + 2^2 + 1^2} = \sqrt{21}$ and $|\nabla g \cdot \mathbf{k}| = 1$.
  - Since the function involves z, we must use the implicit relation to eliminate it. In this case, $z = 1 - 4x - 2y$, so $f(x, y, z) = xz = x - 4x^2 - 2xy$.
  - The desired integral is therefore $\int_0^1 \int_0^1 (x - 4x^2 - 2xy) \cdot \sqrt{21}\,dy\,dx = \int_0^1 (-4x^2)\sqrt{21}\,dx = \boxed{-\dfrac{4}{3}\sqrt{21}}$.
- To compute surface area, we can simply integrate the function 1 on the surface, in exactly the same way that integrating 1 on a plane region gives its area or integrating 1 on a solid region gives its volume.
- Example: Find the area of the portion of the surface $z = 2 - x^2 - y^2$ that lies above the xy-plane.
  - We can rewrite the equation of the surface "implicitly" as $x^2 + y^2 + z - 2 = 0$, so we use the implicit surface formula.
  - The projection of the surface into the xy-plane is the region R on which $2 - x^2 - y^2 \ge 0$, which is the same as $x^2 + y^2 \le 2$, and this describes the disc of radius $\sqrt{2}$ centered at the origin. Since this surface is explicit we do not need to worry about having a vertical tangent plane.
  - We have $\nabla g = \langle 2x, 2y, 1\rangle$ so $||\nabla g|| = \sqrt{4x^2 + 4y^2 + 1}$ and $|\nabla g \cdot \mathbf{k}| = 1$. The desired integral is therefore $\iint_R \sqrt{4x^2 + 4y^2 + 1}\,dy\,dx$, since to calculate surface area we simply integrate the function 1.
  - To evaluate this integral, we change to polar coordinates, since both the region and the function to be integrated will become simpler: the region is $0 \le r \le \sqrt{2}$, $0 \le \theta \le 2\pi$, and the function is $\sqrt{4r^2 + 1}$.
  - Since the area differential in polar is $r\,dr\,d\theta$, we obtain the polar integral $\int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1}\,r\,dr\,d\theta$.
  - To evaluate this new integral, we make (another) substitution $u = 4r^2 + 1$, with $du = 8r\,dr$:

$$\int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2+1}\,r\,dr\,d\theta = \int_0^{2\pi}\int_1^{9} \frac{1}{8}\sqrt{u}\,du\,d\theta = \int_0^{2\pi} \frac{1}{8}\left(\frac{2}{3}u^{3/2}\right)\Big|_{u=1}^{9} d\theta = \int_0^{2\pi} \frac{26}{12}\,d\theta = \boxed{\frac{13\pi}{3}}.$$

  - Remark: Alternatively, we could have parametrized this surface using cylindrical coordinates, as $x = s\cos(t)$, $y = s\sin(t)$, $z = 2 - s^2$ for $0 \le s \le \sqrt{2}$, $0 \le t \le 2\pi$. This would have led us directly to the integral that showed up at the end (with s and t in place of r and θ).

- To find the average value of a function on a surface, we integrate the function on the surface and then divide by the surface area.

- Example: Find the average value of $f(x,y,z) = z$ on the surface S given by the portion of the cone $z = \sqrt{x^2+y^2}$ that lies inside the cylinder $x^2+y^2 = 4$.

  - By using cylindrical coordinates we see that we can parametrize this portion of the cone as $x = s\cos(t)$, $y = s\sin(t)$, $z = s$, for $0 \le s \le 2$ and $0 \le t \le 2\pi$.
  - We then have $\mathbf{r}(s,t) = \langle s\cos(t),\, s\sin(t),\, s\rangle$, so $\dfrac{d\mathbf{r}}{ds} = \langle \cos(t),\, \sin(t),\, 1\rangle$ and $\dfrac{d\mathbf{r}}{dt} = \langle -s\sin(t),\, s\cos(t), 0\rangle$.
  - Then $\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & \sin(t) & 1 \\ -s\sin(t) & s\cos(t) & 0 \end{vmatrix} = \langle -s\cos(t),\, s\sin(t),\, s\rangle$, so the magnitude is given by $\left\|\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right\| = \sqrt{s^2\cos^2(t) + s^2\sin^2(t) + s^2} = s\sqrt{2}$.
  - We also have $f(x,y,z) = z = s$. So $\iint_S z\, d\sigma = \int_0^{2\pi}\int_0^2 s\cdot s\sqrt{2}\, ds\, dt = \int_0^{2\pi} \frac{8}{3}\sqrt{2}\, dt = \dfrac{16\pi\sqrt{2}}{3}$.
  - Also, the surface area is $\iint_S 1\, d\sigma = \int_0^{2\pi}\int_0^2 s\sqrt{2}\, ds\, dt = \int_0^{2\pi} 2\sqrt{2}\, dt = 4\pi\sqrt{2}$.
  - Thus, the average value is $\dfrac{1}{\text{Area}} \iint_S z\, d\sigma = \dfrac{16\pi\sqrt{2}/3}{4\pi\sqrt{2}} = \boxed{\dfrac{4}{3}}$.

- Like with double, triple, and line integrals, we have mass and moment formulas for surface integrals:

- Center of Mass and Moment Formulas (Thin Surface): Given a surface S of variable density $\delta(x,y,z)$ in 3-space:

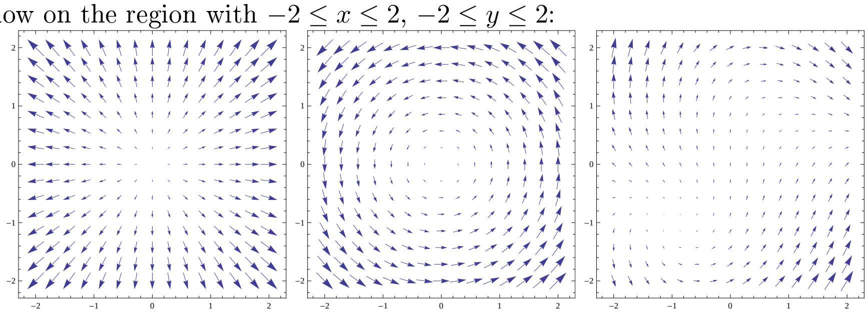
  - The total mass M is given by $M = \iint_S \delta(x,y,z)\, d\sigma$.
  - The x-moment M_{yz} is given by $M_{yz} = \iint_S x\,\delta(x,y,z)\, d\sigma$.
  - The y-moment M_{xz} is given by $M_{xz} = \iint_S y\,\delta(x,y,z)\, d\sigma$.
  - The z-moment M_{xy} is given by $M_{xy} = \iint_S z\,\delta(x,y,z)\, d\sigma$.
  - The center of mass $(\bar{x}, \bar{y}, \bar{z})$ has coordinates $\left(\dfrac{M_{yz}}{M}, \dfrac{M_{xz}}{M}, \dfrac{M_{xy}}{M}\right)$.

- Example: A hill is shaped like the portion of the paraboloid $z = 4 - x^2 - y^2$ with $z \ge 0$, with all coordinates measured in meters. Snow accumulates on the hill such that the density is $\sqrt{17-4z}$ grams per square meter at height z. Find the total amount of snow on the hill.

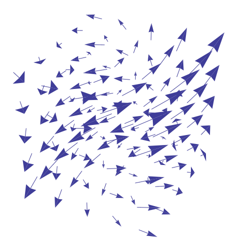
  - We are given the density of snow and want to compute the total mass, which (per the above) is given by the integral $\iint_S \sqrt{17-4z}\, d\sigma$ where S is the surface representing the hill.
  - By using cylindrical coordinates, we can parametrize the hill as $\mathbf{r}(r,\theta) = \left\langle r\cos(\theta),\, r\sin(\theta),\, 4-r^2\right\rangle$, so $\dfrac{\partial \mathbf{r}}{\partial r} = \langle \cos(\theta),\, \sin(\theta),\, -2r\rangle$ and $\dfrac{\partial \mathbf{r}}{\partial \theta} = \langle -r\sin(\theta),\, r\cos(\theta), 0\rangle$.
  - Then $\dfrac{\partial \mathbf{r}}{\partial r} \times \dfrac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & -2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \left\langle 2r^2\cos(\theta),\, 2r^2\sin(\theta),\, r\right\rangle$, so $\left\|\dfrac{\partial \mathbf{r}}{\partial r} \times \dfrac{\partial \mathbf{r}}{\partial \theta}\right\| = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$.
  - We also have $f(x,y,z) = \sqrt{17 - 4(4-r^2)} = \sqrt{4r^2+1}$.
  - Hence the integral becomes $\int_0^{2\pi}\int_0^2 \sqrt{4r^2+1}\cdot r\sqrt{4r^2+1}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 (r+4r^3)\, dr\, d\theta = \int_0^{2\pi} 18\, d\theta = 36\pi$.
  - Thus, there are $\boxed{36\pi \text{ g}}$ of snow on the hill.

4.3 Vector Fields, Work, Circulation, Flux

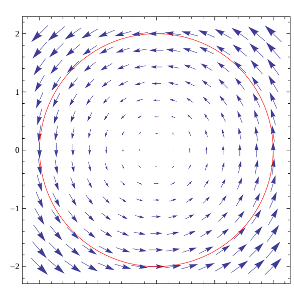
- Definition: A vector field in the plane is a function $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ that associates a vector to each point in the plane. A vector field in 3-space is a function $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ that associates a vector to each point in 3-space.
  - One vector field we have already encountered is the vector field associated to the gradient of a function $f(x, y)$ or $f(x, y, z)$: for example, if $f(x, y) = x^2 + xy$, then $\nabla f(x, y) = \langle 2x + y, x\rangle$.
- To represent a vector field visually, we choose some (nice) collection of points (generally in a grid) and draw the vectors corresponding to those points as arrows pointing in the appropriate direction and with the appropriate length.
  - Example: The three vector fields $\mathbf{F}(x, y) = \langle x, y\rangle$, $\mathbf{G}(x, y) = \langle -y, x\rangle$, and $\mathbf{H}(x, y) = \left\langle x + y^2, 2 - 2xy\right\rangle$ are plotted below on the region with $-2 \le x \le 2$, $-2 \le y \le 2$:



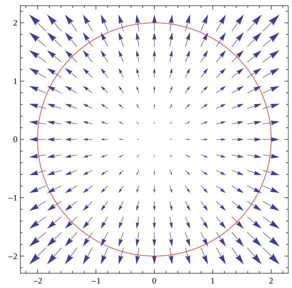
  - We can also produce these plots for 3-dimensional vector fields, but the diagrams tend to be quite cluttered; here is such a diagram for $\mathbf{F}(x, y, z) = \langle x, z - y, x + y\rangle$:



- We can think of a vector field as describing the flow of an incompressible fluid through space: the vector $\mathbf{F}(x, y)$ at any point (x, y) gives the direction and velocity of the fluid's flow there.
- In this context, if we have a particle that travels along some given path $\mathbf{r}(t)$ through the fluid, we might like to know how much work the fluid does on the particle, or (essentially equivalently) how much the fluid is pushing the particle along its path. This is the central idea behind work integrals and circulation integrals.
  - Intuitively, we see that the more the vector field $\mathbf{F}$ aligns with the tangent vector $\mathbf{T}$ to the particle's path, the more work it does.
  - In the picture, a particle moving counterclockwise around the circle will be pushed along its path by the vector field:



- Alternatively, if we have a particle traveling along a path, we could also ask: how much is the fluid pushing the particle off of the path? This is the central idea behind a flux integral.
  - Another way of thinking about this is to imagine the path as being a thin membrane, and asking how much fluid is passing across the membrane.
  - Here, we see that more fluid is flowing across the membrane if the vector field $\mathbf{F}$ aligns with the normal vector $\mathbf{N}$ to the particle's path:



- We can also formulate these ideas in 3-dimensional space: the ideas of circulation and work remain the same, but the notion of flux requires a surface for the fluid to flow across.

4.3.1 Circulation and Work Integrals

- To compute the circulation of a vector field along a curve, we want to integrate the quantity measuring how much the vector field is aligning with the path of motion along the curve.
- Definition: The (counterclockwise) circulation (or flow) of the vector field $\mathbf{F}$ along the curve C is defined to be $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{T}$ is the unit tangent to the curve.
  - What this says is: the circulation is given by integrating the dot product function $f(t) = \mathbf{F}(x(t),\, y(t)) \cdot \mathbf{T}(t)$ along the curve C. In order to evaluate the integral as written, we would need to parametrize C, find the unit tangent vector $\mathbf{T}(t)$ to the curve, and then integrate the dot product $\mathbf{F}(x(t),\, y(t)) \cdot \mathbf{T}(t)$ along the curve.
  - We would like to see if there is a simpler way, so let us suppose that $\mathbf{F}(x, y) = \langle P, Q\rangle$, where P and Q are functions of x and y, and say C is parametrized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ from $t = a$ to $t = b$.
  - Then $\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{||\mathbf{v}(t)||} = \dfrac{\left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt} \right\rangle}{||\mathbf{v}(t)||}$, so $\mathbf{F} \cdot \mathbf{T} = \dfrac{\langle P, Q\rangle \cdot \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt} \right\rangle}{||\mathbf{v}(t)||} = \dfrac{P\,\dfrac{dx}{dt} + Q\,\dfrac{dy}{dt}}{||\mathbf{v}(t)||}$.
  - We can then write $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \dfrac{P\,\dfrac{dx}{dt} + Q\,\dfrac{dy}{dt}}{||\mathbf{v}(t)||} \cdot ||\mathbf{v}(t)||\; dt = \int_a^b \left[P\,\dfrac{dx}{dt} + Q\,\dfrac{dy}{dt}\right] dt$.
  - Thus, the circulation integral can be written more explicitly as $\int_a^b \left[P\,\dfrac{dx}{dt} + Q\,\dfrac{dy}{dt}\right] dt$, where P, Q have been rewritten as functions of t. Note that this expression is also equal to $\int_C P\, dx + Q\, dy$.
  - We can also pose essentially the same definition for a curve in 3-space, and we obtain an analogous formula: if $\mathbf{F} = \langle P, Q, R\rangle$, then the circulation can be computed as $\int_a^b \left[P\,\dfrac{dx}{dt} + Q\,\dfrac{dy}{dt} + R\,\dfrac{dz}{dt}\right] dt$.
  - Terminology Note: Some authors reserve the term "circulation" for closed curves, and use "flow" to refer to the general case. This terminology can be somewhat confusing given that there is also a "flux" integral, and the words "flux" and "flow" (in non-technical settings) are synonyms.
- Example: Find the circulation of the vector field $\mathbf{G}(x, y) = \langle -y, x\rangle$ around a path that winds once counterclockwise around the unit circle.
  - We can parametrize the path as $x = \cos t$, $y = \sin t$ for $0 \le t \le 2\pi$.
  - Thus, $P = -y = -\sin t$ and $Q = x = \cos t$, and also $\dfrac{dx}{dt} = -\sin t$ and $\dfrac{dy}{dt} = \cos t$.

- So, the circulation is $\int_a^b \left(P\,\frac{dx}{dt}+Q\,\frac{dy}{dt}\right)\,dt = \int_0^{2\pi}\left((-\sin t)(-\sin t)+(\cos t)(\cos t)\right)\,dt = \int_0^{2\pi} 1\,dt = \boxed{2\pi}$.

- Example: Find the circulation of the vector field $\mathbf{F}(x,y,z)=\langle 2xy, xz, y\rangle$ along the line segment from $(0,1,0)$ to $(2,2,2)$.
  - We can parametrize the path as $x=2t$, $y=1+t$, $z=2t$ for $0\le t\le 1$.
  - Thus, $P=2xy=4t+4t^2$, $Q=xy=4t^2$, and $R=1+t$.
  - So, the circulation is $\int_a^b \left(P\,\frac{dx}{dt}+Q\,\frac{dy}{dt}+R\,\frac{dz}{dt}\right)\,dt = \int_0^1 [(4t+4t^2)\cdot 2+4t^2\cdot 1+(1+t)\cdot 2]\,dt = \int_0^1 (2+10t+12t^2)\,dt = \boxed{11}$.

- We can also pose a similar definition for the work done by a vector field on a particle:

- Definition: The work performed on a particle by a vector field $\mathbf{F}$ as the particle travels along a curve C is $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$.
  - Note that the work integral has the same form as the circulation integral.
  - Notation: The "vector differential" $d\mathbf{r}$ is defined as $d\mathbf{r}=\langle dx, dy\rangle$ in the plane and as $d\mathbf{r}=\langle dx,\,dy,\,dz\rangle$ in 3-space.
  - Then $\mathbf{F}\cdot d\mathbf{r} = P\,dx+Q\,dy$, so the work integral is $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx+Q\,dy = \int_a^b \left[P\,\frac{dx}{dt}+Q\,\frac{dy}{dt}\right]\,dt$ in the plane, or as $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx+Q\,dy+R\,dz = \int_a^b \left[P\,\frac{dx}{dt}+Q\,\frac{dy}{dt}+R\,\frac{dz}{dt}\right]\,dt$ in 3-space.

- Example: Find the work done by the vector field $\mathbf{F}(x,y,z)=\langle 2x+z,\,yz,\,xy\rangle$ on a particle traveling along the path $\mathbf{r}(t)=\langle t,\,t^2,\,2t\rangle$ from $t=0$ to $t=1$.
  - We have $P=2x+z=3t$, $Q=yz=2t^3$, and $R=xy=t^3$. Also, $\frac{dx}{dt}=1$, $\frac{dy}{dt}=2t$, and $\frac{dz}{dt}=2$.
  - Therefore, the work is $\int_a^b \left(P\,\frac{dx}{dt}+Q\,\frac{dy}{dt}+R\,\frac{dz}{dt}\right)\,dt = \int_0^1 \left[(3t)(1)+(2t^3)(2t)+(t^3)(2)\right]\,dt = \int_0^1 (3t+4t^4+2t^3) = \boxed{\frac{14}{5}}$.

4.3.2 Flux Across a Curve

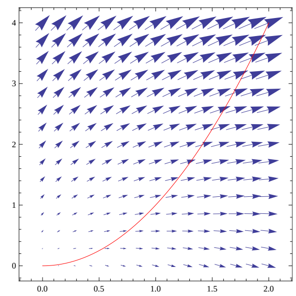
- To compute the flux of a vector field across a curve, we want to integrate the quantity measuring how much the vector field is moving in the direction perpendicular to the curve.

- Definition: The flux of the vector field $\mathbf{F}$ across the curve C is $\int_C \mathbf{F}\cdot\mathbf{N}\,ds$, where $\mathbf{N}$ is the unit normal to the curve.
  - As with the circulation integral, we would like an easier way to evaluate the flux integral.
  - If $\mathbf{F}(x,y)=\langle P,Q\rangle$ and C is parametrized by $\mathbf{r}(t)=\langle x(t),y(t)\rangle$ from $t=a$ to $t=b$, after some algebra we can calculate that $\mathbf{N}(t)=\frac{1}{||\mathbf{v}(t)||}\left\langle \frac{dy}{dt},\,-\frac{dx}{dt}\right\rangle$. (At the very least, it is easy to observe that this is a unit vector that is orthogonal to $\mathbf{T}$.)
  - Then $\mathbf{F}\cdot\mathbf{N} = \dfrac{\langle P,\,Q\rangle\cdot\left\langle \frac{dy}{dt},\,-\frac{dx}{dt}\right\rangle}{||\mathbf{v}(t)||} = \dfrac{P\,\frac{dy}{dt}-Q\,\frac{dx}{dt}}{||\mathbf{v}(t)||}$.
  - Plugging this in gives $\int_C \mathbf{F}\cdot\mathbf{N}\,ds = \int_a^b \dfrac{P\,\frac{dy}{dt}-Q\,\frac{dx}{dt}}{||\mathbf{v}(t)||}\cdot ||\mathbf{v}(t)||\,dt = \int_a^b \left[P\,\frac{dy}{dt}-Q\,\frac{dx}{dt}\right]\,dt$.
  - Thus, the flux integral can be written more explicitly as $\int_C P\,dy-Q\,dx = \int_a^b \left[P\,\frac{dy}{dt}-Q\,\frac{dx}{dt}\right]\,dt$.

- Note: The flux integral as defined here only makes sense for curves in the plane. In 3-dimensional space, the corresponding notion requires a surface integral, since a "membrane" will be a surface, rather than a curve.

- Example: Find the flux of the vector field $\mathbf{G}(x,y) = \langle x, y\rangle$ across a path that winds once counterclockwise around the unit circle.
  - We can parametrize the path as $x = \cos t$, $y = \sin t$ for $0 \le t \le 2\pi$.
  - Thus, $P = x = \cos t$ and $Q = y = \sin t$, and also $\dfrac{dx}{dt} = -\sin t$ and $\dfrac{dy}{dt} = \cos t$.
  - So Flux $= \int_a^b \left(P\dfrac{dy}{dt} - Q\dfrac{dx}{dt}\right) dt = \int_0^{2\pi} \left((\cos t)(\cos t) - (\sin t)(-\sin t)\right) dt = \int_0^{2\pi} 1\,dt = \boxed{2\pi}$.

- Example: For the vector field $\mathbf{F}(x,y) = \langle 2x+y, 2y-x\rangle$, find the flux across, and circulation along, the portion of the curve $\mathbf{r}(t) = \langle t, t^2\rangle$ between $(0,0)$ and $(2,4)$.
  - Here is a plot of the vector field, along with the curve:



  - From the picture, we would expect the circulation and flux to be roughly equal, since the vector field makes roughly a 45-degree angle with the path near the end.
  - The parametrization given says $x = t$ and $y = t^2$, so that $P = 2x + y = 2t + t^2$ and $Q = 2y - x = 2t^2 - t$. Also, the start is $t = 0$ and the end is $t = 2$.
  - Then the circulation is $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \left(P\dfrac{dx}{dt} + Q\dfrac{dy}{dt}\right) dt = \int_0^2 \left((2t+t^2)\cdot 1 + (2t^2-t)\cdot 2t\right) dt = \int_0^2 \left(4t^3 - t^2 + 2t\right) dt = \left(t^4 - \dfrac{1}{3}t^3 + t^2\right)\Big|_{t=0}^{2} = \boxed{\dfrac{52}{3}}$.
  - The flux is $\int_C \mathbf{F}\cdot\mathbf{N}\,ds = \int_a^b \left(P\dfrac{dy}{dt} - Q\dfrac{dx}{dt}\right) dt = \displaystyle\int_0^2 \left((2t+t^2)\cdot 2t - (2t^2-t)\cdot 1\right) dt = \int_0^2 \left(2t^3 + 2t^2 + 2t\right) dt = \left(\dfrac{1}{2}t^4 + \dfrac{2}{3}t^3 + t^2\right)\Big|_{t=0}^{2} = \boxed{\dfrac{52}{3}}$.
  - Indeed, we see that the flux and circulation are roughly (and exactly) equal.

4.3.3 Flux Across a Surface

- In 3-space, the notion of circulation along a curve remains essentially the same as in the plane. However, in order to make sense of flux in 3-space, we must instead talk about flux through a surface rather than through a curve. This requires us to use a surface integral to measure how much the vector field is flowing across the surface:

- Definition: The (normal) flux of the vector field $\mathbf{F}$ across the surface S is $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$, where $\mathbf{n}$ is the outward unit normal to the surface.
  - Remark: The integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ computes the flux through the surface in the direction of the "outward normal". It is also possible to ask about flux in the direction of a particular unit vector $\mathbf{u}$; the integral in that case is $\iint_S \mathbf{F}\cdot\mathbf{u}\,d\sigma$, instead. In general, when it is not specified what type of flux integral is meant, the "flux in the direction of the outward normal" is intended.

- Recall that the normal vector to a surface is orthogonal to the tangent plane (it is in fact the normal vector to the tangent plane as we defined it earlier). When speaking of a unit normal to a surface we will use a lowercase $\mathbf{n}$, to keep the notation different from the unit normal $\mathbf{N}$ to a curve (which is an uppercase $\mathbf{N}$).
- If S is an implicit surface $g(x,y,z)=c$, then a normal vector is given by the gradient ∇g, so we get a unit normal vector $\mathbf{n} = \nabla g / ||\nabla g||$.
- If S is parametrized by $\mathbf{r}(s,t) = \langle x(s,t),\, y(s,t),\, z(s,t)\rangle$, then a normal vector is given by the cross product $\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}$, so we get a unit normal vector $\mathbf{n} = \left(\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right) \Big/ \left|\left|\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right|\right|$.
- Important Warning: If we scale the implicit equation by -1, or write the factors of the cross product in the opposite order, the resulting normal vector $\mathbf{n}$ is multiplied by -1. To remedy this ambiguity, we must always specify which of these two possible orientations of the normal vector we intend. You should always check to ensure that the normal vector is pointing in the correct direction: typical conventions are for it to be pointing "outward" or "upward".

- By plugging these expressions into the surface integral formula, we obtain explicit formulas for the outward normal flux across a surface S:
  - If S is parametrized by $\mathbf{r}(s,t) = \langle x(s,t),\, y(s,t),\, z(s,t)\rangle$, then the outward normal flux across S is equal to $\iint_S \mathbf{F} \cdot \left(\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right) ds\, dt$, provided that $\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}$ is the outward-pointing normal vector to the surface. (Conveniently, the unpleasant part of the surface-area differential cancels out the normalization in the unit normal vector.)
  - If S is defined implicitly by $f(x,y,z)=c$ and R is the projection of S in the xy-plane, then the outward normal flux across S is equal to $\iint_R \dfrac{\mathbf{F}\cdot\nabla g}{|\nabla g \cdot \mathbf{k}|}\, dy\, dx$. Note here that the denominator term $\nabla g \cdot \mathbf{k}$ is simply the partial derivative g_z.
  - Depending on the description of the surface, either of these particular approaches (via a parametrization or as an implicit surface) may be more convenient for computing a flux integral.

- Example: Find the outward flux of the vector field $\mathbf{F} = \langle xz^2, yz^2, x^3 e^y\rangle$ through the portion of the cylinder $x^2+y^2=4$ that lies between the planes $z=-1$ and $z=1$.
  - From cylindrical coordinates, we can parametrize the cylinder as $\mathbf{r}(s,t) = \langle 2\cos t, 2\sin t, s\rangle$, where the desired portion corresponds to $-1 \le s \le 1$ and $0 \le t \le 2\pi$.
  - Then $\dfrac{\partial \mathbf{r}}{\partial t} = \langle -2\sin t,\, 2\cos t,\, 0\rangle$ and $\dfrac{\partial \mathbf{r}}{\partial s} = \langle 0,0,1\rangle$, so $\dfrac{\partial \mathbf{r}}{\partial t} \times \dfrac{\partial \mathbf{r}}{\partial s} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin t & 2\cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2\cos t, 2\sin t, 0\rangle$.
  - This is indeed an outward-pointing normal vector since it is the vector pointing from $(0,0,s)$ to the point $\mathbf{r}(s,t) = (2\cos t, 2\sin t, s)$ on the surface.
  - Then $\mathbf{F} \cdot \left(\dfrac{\partial \mathbf{r}}{\partial t} \times \dfrac{\partial \mathbf{r}}{\partial s}\right) = \left\langle 2s^2\cos t, 2s^2 \sin t, (2\cos t)^3 e^{2\sin t}\right\rangle \cdot \langle 2\cos t,\, 2\sin t,\, 0\rangle = 4s^2\cos^2 t + 4s^2 \sin^2 t = 4s^2$.
  - The flux integral is thus $\int_0^{2\pi}\int_{-1}^{1} 4s^2\, ds\, dt = \int_0^{2\pi} \dfrac{8}{3}\, dt = \boxed{\dfrac{16\pi}{3}}$.

- Example: Find the outward flux of the vector field $\mathbf{F} = \langle x-z,\, y, x+z\rangle$ through the portion of the sphere $x^2+y^2+z^2=4$ that lies above the plane $z=1$.
  - We use the formula for flux across an implicit surface.
  - On the sphere, $z=1$ corresponds to $x^2+y^2=3$, and as z increases to 2, the value of x^2+y^2 decreases to 0. Thus the projection of the surface into the xy-plane is the region $R: x^2+y^2 \le 3$.
  - We have $\nabla g = \langle 2x, 2y, 2z\rangle$, so $\dfrac{\mathbf{F}\cdot\nabla g}{|\nabla g\cdot\mathbf{k}|} = \dfrac{2x^2 - 2xz + 2y^2 + 2xz + 2z^2}{2z} = \dfrac{4}{\sqrt{4-x^2-y^2}}$.

- The flux integral is therefore given by $\iint_R \dfrac{4}{\sqrt{4-x^2-y^2}}\,dy\,dx$. We will evaluate this integral using polar coordinates.
- In polar coordinates, the region is $0 \le r \le \sqrt{3}$ and $0 \le \theta \le 2\pi$, so the integral is $\int_0^{2\pi}\int_0^{\sqrt{3}} \dfrac{4}{\sqrt{4-r^2}}\,r\,dr\,d\theta$.
- Substituting $u = 4 - r^2$ in the inner integral gives $\int_0^{2\pi}\int_0^{\sqrt{3}} \dfrac{4}{\sqrt{4-r^2}}\,r\,dr\,d\theta = \int_0^{2\pi}\int_1^0 -\dfrac{2}{\sqrt{u}}\,du\,d\theta = \int_0^{2\pi} 4\,d\theta = \boxed{8\pi}$.
- Alternatively, we could have observed that for a sphere of radius ρ centered at the origin, the outward unit normal vector is $\mathbf{n} = \dfrac{1}{\rho}\langle x, y, z\rangle$.
- The desired integral is therefore $\iint_S \dfrac{1}{2}\langle x, y, z\rangle \cdot \langle x - z, y, x + z\rangle\,d\sigma = \iint_S \dfrac{1}{2}(x^2+y^2+z^2)\,d\sigma = \iint_S 2\,d\sigma$.
- This is twice the surface area of S, which we could compute (using a simpler surface integral) to be 4π, meaning that the desired flux is again $\boxed{8\pi}$.

- Example: Find the outward flux of the vector field $\mathbf{F} = \langle x, y, z\rangle$ through the sphere $x^2+y^2+z^2 = 9$.
  - Using spherical coordinates, we can parametrize the sphere as $\mathbf{r}(s,t) = \langle 3\sin s\cos t, 3\sin s\sin t, 3\cos s\rangle$ for $0 \le s \le \pi$ and $0 \le t \le 2\pi$.
  - Then $\dfrac{\partial \mathbf{r}}{\partial t} = \langle -3\sin s\sin t, 3\sin s\cos t, 0\rangle$ and $\dfrac{\partial \mathbf{r}}{\partial s} = \langle 3\cos s\cos t, 3\cos s\sin t, -3\sin s\rangle$, so $\dfrac{\partial \mathbf{r}}{\partial t} \times \dfrac{\partial \mathbf{r}}{\partial s} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\sin s\sin t & 3\sin s\cos t & 0 \\ 3\cos s\cos t & 3\cos s\sin t & -3\sin s \end{vmatrix} = \left\langle -9\sin^2 s\cos t, -9\sin^2 s\sin t, -9\sin s\cos s\right\rangle$.
  - This is not an outward-pointing normal vector, since it is $-3\sin s$ times the position vector $\mathbf{r}(s,t)$, so we must scale it by -1.
  - Then $\mathbf{F}\cdot -\left(\dfrac{\partial \mathbf{r}}{\partial t} \times \dfrac{\partial \mathbf{r}}{\partial s}\right) = \langle 3\sin s\cos t, 3\sin s\sin t, 3\cos s\rangle \cdot \left\langle 9\sin^2 s\cos t, 9\sin^2 s\sin t, 9\sin s\cos s\right\rangle = 27\sin^3 s\cos^2 t + 27\sin^3 s\sin^2 t + 27\sin s\cos^2 s = 27\sin s$.
  - The flux integral is thus $\int_0^{2\pi}\int_0^{\pi} 27\sin s\,ds\,dt = \int_0^{2\pi} 54\,dt = \boxed{108\pi}$.

4.4 Conservative Vector Fields, Path-Independence, and Potential Functions

- If we have a vector field $\mathbf{F}(x,y)$ and two different paths C_1 and C_2 between the same two points, we might wonder if there is any relation between the work integrals $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$.

- Example: For the fields $\mathbf{F}(x,y) = \langle y, x\rangle$ and $\mathbf{G}(x,y) = \left\langle y^2, x\right\rangle$ evaluate the work integrals from $(0,0)$ to $(1,1)$ along the three different paths $C_1 : (x,y) = (t,t)$, $C_2 : (x,y) = (t^3, t^2)$, and $C_3 : (x,y) = (t^7, t^{10})$, for $0 \le t \le 1$.
  - Along C_1 we have $\mathbf{F} = \langle t, t\rangle$, $\mathbf{G} = \left\langle t^2, t\right\rangle$, $\dfrac{dx}{dt} = 1$, and $\dfrac{dy}{dt} = 1$.
  - Then $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 [t\cdot 1 + t\cdot 1]\,dt = \boxed{1}$, and $\int_{C_1} \mathbf{G}\cdot d\mathbf{r} = \int_0^1 \left[t^2\cdot 1 + t\cdot 1\right]dt = \boxed{\dfrac{5}{6}}$.
  - Along C_2 we have $\mathbf{F} = \left\langle t^2, t^3\right\rangle$, $\mathbf{G} = \left\langle t^4, t^3\right\rangle$, $\dfrac{dx}{dt} = 3t^2$, and $\dfrac{dy}{dt} = 2t$.
  - Then $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left[t^2\cdot 3t^2 + t^3\cdot 2t\right]dt = \boxed{1}$, and $\int_{C_2} \mathbf{G}\cdot d\mathbf{r} = \int_0^1 \left[t^4\cdot 3t^2 + t^3\cdot 2t\right]dt = \boxed{\dfrac{29}{35}}$.
  - Along C_3 we have $\mathbf{F} = \left\langle t^{10}, t^7\right\rangle$, $\mathbf{G} = \left\langle t^{20}, t^7\right\rangle$, $\dfrac{dx}{dt} = 7t^6$, and $\dfrac{dy}{dt} = 10t^9$.
  - Then $\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left[t^{10}\cdot 7t^6 + t^7\cdot 10t^9\right]dt = \boxed{1}$, and $\int_{C_3} \mathbf{G}\cdot d\mathbf{r} = \int_0^1 \left[t^{30}\cdot 7t^6 + t^7\cdot 10t^9\right]dt = \boxed{\dfrac{389}{459}}$.

- Observe that for $\mathbf{F}$, all three paths give the same value, while for $\mathbf{G}$, each path gives a different value.

- We would like to understand what about $\mathbf{F}$ in the example above seems to cause it to do the same amount of work regardless of the path we chose.

- Definition: A vector field $\mathbf{F}$ is conservative on a region R if, for any two paths C_1 and C_2 (inside R) from P_1 to P_2, it is true that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. In other words, $\mathbf{F}$ is conservative if any two paths with the same endpoints yield the same work integral.
  - Equivalent to the above definition is the following: $\mathbf{F}$ is conservative on a region R if, for any closed curve C in R, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$. (A closed curve is one whose start and end points are the same.)
  - Notation: For a line integral around a closed curve, we often use the notation $\oint_C$, the circle being a suggestive example of a closed curve.
  - These two statements are equivalent because, if C_1 and C_2 are two paths from P_1 to P_2, then we can construct a closed path C by following C_1 from P_1 to P_2 and then following C_2 from P_2 back to P_1. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, and so the left-hand side is zero if and only if the right-hand side is zero.

- It turns out that we can give a simple but very useful criterion for when a vector field is conservative:

- Theorem (Fundamental Theorem of Calculus for Line Integrals): The vector field $\mathbf{F}$ is conservative on a simply-connected region R if and only if there exists a function U, called a potential function for $\mathbf{F}$, such that $\mathbf{F} = \nabla U$. If such a function U exists, then $\int_a^b \mathbf{F} \cdot d\mathbf{r} = U(b) - U(a)$ along any path from a to b.
  - Notice the similarity of the statement $\int_a^b \mathbf{F} \cdot d\mathbf{r} = U(b) - U(a)$ to the Fundamental Theorem of Calculus, which relates the integral of a derivative of a function to its values at the endpoints of a path.
  - Technical Note: The term "simply-connected" is a technical requirement needed for the proof of the theorem: intuitively, a simply-connected region consists of a single piece that does not have any "holes" in it. More rigorously, it means that the region is connected (contains only one "piece") and that if we take any closed loop in the region, we can shrink it to a point without leaving the region. The disc $x^2 + y^2 \le 4$ is simply-connected, whereas the annulus $1 \le x^2 + y^2 \le 4$ is not.
  - The full proof is not especially enlightening. We will instead show one direction of the proof.
  - Proof (Reverse Direction in 3-Space): Suppose that $\mathbf{F} = \nabla U = \left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right\rangle$.
  - By the (multivariable) Chain Rule, if C is the path with $x = x(t)$, $y = y(t)$, and $z = z(t)$ for $a \le t \le b$, then $\frac{dU}{dt} = \frac{\partial U}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial U}{\partial z} \cdot \frac{dz}{dt}$.
  - Now we can write

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt \\ &= \int_a^b \left[\frac{\partial U}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial U}{\partial z} \cdot \frac{dz}{dt} \right] dt \\ &= \int_a^b \left[\frac{dU}{dt} \right] dt = U(\mathbf{r}(b)) - U(\mathbf{r}(a)) \end{aligned}$$

    where we used the Fundamental Theorem of Calculus for the last step.
  - Notice that this expression does not depend on C: it only involves the potential function U and the two endpoints $\mathbf{r}(b)$ and $\mathbf{r}(a)$. Hence we see that the integral is independent of the path, so $\mathbf{F}$ is conservative.

- If we can see that a vector field is conservative, then it is very easy to compute work integrals: we just need to find a potential function for the vector field.

- Example: Find the work done by the vector field $\mathbf{F}(x, y) = \langle 2x + y, x \rangle$ on a particle traveling along the path $\mathbf{r}(t) = \langle -2\cos(\pi e^t), \tan^{-1}(t) \rangle$ from $t = 0$ to $t = 1$.

- If we try to set up the integral directly using the parametrization, it will be rather unpleasant.
- However, this vector field is conservative: it is not hard to see that for $U(x, y) = x^2 + xy$, we have $\nabla U = \langle 2x + y, x\rangle = \mathbf{F}$.
- By the Fundamental Theorem of Calculus for line integrals, the work done by the vector field is then simply the value of $U(\mathbf{r}(1)) - U(\mathbf{r}(0))$.
- Since $\mathbf{r}(1) = \langle 2, \pi/4\rangle$ and $\mathbf{r}(0) = \langle -2, 0\rangle$, the work is $U(2, \pi/4) - U(-2, 0) = \boxed{\dfrac{\pi}{2}}$.

- We would like to be able to determine easily whether a given vector field is conservative. To do this, we require a preliminary definition:

- Definition: If $\mathbf{F} = \langle P, Q, R\rangle$ then the curl of $\mathbf{F}$ is defined to be the vector field $\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \left\langle \dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}, \dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}, \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right\rangle = \langle R_y - Q_z,\, P_z - R_x,\, Q_x - P_y\rangle$.
  - Example: If $\mathbf{F} = \langle 3x^2y, xyz, e^{xy}\rangle$ then $\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \boxed{\langle xe^{xy} - xy,\, -ye^{xy},\, yz - 3x^2\rangle}$.
  - If $\mathbf{F} = \langle P, Q\rangle$ is a vector field in the plane then we define the curl of $\mathbf{F}$ to be the curl of the vector field $\langle P, Q, 0\rangle$: namely, $\left\langle 0, 0, \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right\rangle$.
  - Since this vector only has one nonzero component, some authors define the curl of a vector field in the plane to be the scalar quantity $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$. We will not do this: for us, the curl of a vector field will always be a new vector field.

- The curl of a vector field determines whether or not it is conservative:

- Theorem (Zero Curl Implies Conservative): A vector field on a simply-connected region in the plane or in 3-space is conservative if and only if its curl is zero. More explicitly, the vector field $\mathbf{F} = \langle P, Q\rangle$ is conservative on a simply-connected region R in the plane if and only if $P_y = Q_x$, and the vector field $\mathbf{F} = \langle P, Q, R\rangle$ is conservative on a simply-connected region D in 3-space if and only if $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$.
  - It is fairly easy to see why we need the equality of the derivatives of the components: if $\mathbf{F} = \langle P, Q\rangle = \nabla U$ then $P = U_x$ and $Q = U_y$, so by the equality of mixed partial derivatives, we see that $P_y = U_{xy} = U_{yx} = Q_x$.
  - The three necessary equalities when $\mathbf{F} = \langle P, Q, R\rangle$ follow in the same way: if $\mathbf{F} = \nabla U$ then $P = U_x$, $Q = U_y$, and $R = U_z$, so $P_y = U_{xy} = U_{yx} = Q_x$, $P_z = U_{xz} = U_{zx} = R_x$, and $Q_z = U_{yz} = U_{zy} = R_y$.
  - The converse statement (that zero curl implies the field is conservative) is more difficult, and we omit the verification.

- The two theorems give us an effective procedure for determining whether a field is conservative: we first check whether its curl is zero, and then (if it is) we can try to find a potential function by computing antiderivatives.

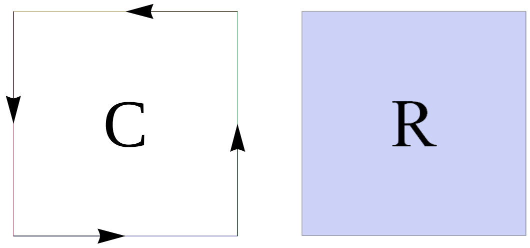
- Example: Determine whether $\mathbf{F}(x, y) = \langle x^2 + y, x + y^2\rangle$ is conservative, and if so, find a potential function.
  - For $\mathbf{F}$, we see $\dfrac{\partial}{\partial y}\left[x^2 + y\right] = 1 = \dfrac{\partial}{\partial x}\left[x + y^2\right]$, so the field is $\boxed{\text{conservative}}$.
  - To find a potential function U with $\nabla U = \mathbf{F}$, we need to find U such that $U_x = x^2 + y$ and $U_y = x + y^2$.
  - Taking the antiderivative of $U_x = x^2 + y$ with respect to x yields $U = \dfrac{1}{3}x^3 + xy + f(y)$, for some function $f(y)$.
  - To find $f(y)$ we differentiate: $U_y = x + f'(y)$, so we get $f'(y) = y^2$ so $f(y) = \dfrac{1}{3}y^3$. (Plus an arbitrary constant, but we can ignore it.)
  - Thus we see that a potential function for $\mathbf{F}$ is $U(x, y) = \boxed{\dfrac{1}{3}x^3 + xy + \dfrac{1}{3}y^3}$.

- Example: Determine whether $\mathbf{G}(x,y) = \langle x+y^2, x^2+y \rangle$ is conservative, and if so, find a potential function.
  - For $\mathbf{G}$, we see $\frac{\partial}{\partial y}\left[x+y^2\right] = 2y \neq 2x = \frac{\partial}{\partial x}\left[x^2+y\right]$, so the field is $\boxed{\text{not conservative}}$.
- Example: Determine whether $\mathbf{H}(x,y,z) = \langle y+2z, x+3z, 2x+3y \rangle$ is conservative, and if so, find a potential function.
  - For $\mathbf{H}$, we have $\frac{\partial}{\partial y}[y+2z] = 1 = \frac{\partial}{\partial x}[x+3z]$, $\frac{\partial}{\partial z}[y+2z] = 2 = \frac{\partial}{\partial x}[2x+3y]$, and $\frac{\partial}{\partial z}[x+3z] = 3 = \frac{\partial}{\partial y}[2x+3y]$, so the field is $\boxed{\text{conservative}}$.
  - To find a potential function U with $\nabla U = \mathbf{H}$, we need to find U such that $U_x = y+2z$, $U_y = x+3z$, and $U_z = 2x+3y$.
  - Taking the antiderivative of $U_x = y+2z$ with respect to x yields $U = xy+2xz+f(y,z)$, for some function $f(y,z)$.
  - To find $f(y,z)$ we differentiate: $x+f_y = x+3z$ and $2x+f_z = 2x+3y$, so $f_y = 3z$ and $f_z = 3y$. Repeating the process yields $f = 3yz+g(z)$, where $g'(z) = 0$.
  - Thus we see that a potential function for $\mathbf{H}$ is $U(x,y,z) = \boxed{xy+2xz+3yz}$.
- If we can find a potential function for a conservative vector field, then (as we saw above) we can use it to compute work integrals.
- Example: If $\mathbf{F} = \left\langle x^3+4x^3\sin y\,\sin z+y^2z,\ 2xyz+y+x^4\cos y\,\sin z,\ z^3+x^4\sin y\,\cos z+xy^2 \right\rangle$, find the work done by $\mathbf{F}$ on a particle that travels along the curve $C : \mathbf{r}(t) = \left\langle \sin(\pi t),\ t\sqrt{t+3},\ 2t^3+2 \right\rangle$ for $0 \le t \le 1$.
  - In theory we could compute the work integral using the parametrization of the path, but this seems quite unpleasant. Instead, we will check whether this vector field is conservative: then determining the answer only requires us to find the potential function of the field.
  - We have $P_y = 4x^3\cos y\,\sin z+2yz$ and $Q_x = 2yz+4x^3\cos y\,\sin z$ so they are equal.
  - We have $P_z = 4x^3\sin y\,\cos z+y^2$ and $R_x = 4x^3\sin y\,\cos z+y^2$ so they are also equal.
  - Finally we have $Q_z = 2xy+x^4\cos y\,\cos z$ and $R_y = 4x^3\cos y\,\cos z+2xy$, and these are also equal. Thus, the field is conservative.
  - To find a potential function U with $\mathbf{F} = \nabla U = \langle U_x, U_y, U_z \rangle$:
    * We know $U_x = x^3+4x^3\sin y\,\sin z+y^2z$ so taking the antiderivative with respect to x yields $U = \frac{1}{4}x^4+x^4\sin y\,\sin z+xy^2z+C(y,z)$.
    * We then see $U_y = x^4\cos y\,\sin z+2xyz+C_y(y,z)$ must equal $2xyz+y+x^4\cos y\,\sin z$ so we see $C_y = y$. Then taking the antiderivative with respect to y yields $C(y,z) = \frac{1}{2}y^2+D(z)$.
    * We now have $U = \frac{1}{4}x^4+x^4\sin y\,\sin z+xy^2z+\frac{1}{2}y^2+D(z)$. Then $U_z = x^4\sin y\,\cos z+xy^2+D'(z)$ must equal $z^3+x^4\sin y\,\cos z+xy^2$ so we see $D'(z) = z^3$ so we can take $D(z) = \frac{1}{4}z^4$.
  - We conclude that a potential function for $\mathbf{F}$ is $U(x,y,z) = \frac{1}{4}x^4+x^4\sin y\,\sin z+xy^2z+\frac{1}{2}y^2+\frac{1}{4}z^4$.
  - Then the desired work integral is equal to $U(0,2,4)-U(0,0,2) = \boxed{62}$.

4.5 Green's Theorem

- Green's Theorem is a 2-dimensional version of the Fundamental Theorem of Calculus that relates a line integral of a function around a closed curve C to the double integral of a related function over the region R that is enclosed by the curve C.

- Theorem (Green's Theorem): If C is a simple closed rectifiable curve oriented counterclockwise, and R is the region it encloses, then for any continuously differentiable functions $P(x, y)$ and $Q(x, y)$, $\displaystyle\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d$
    - Here is an example of a curve C and its corresponding region R:



    - Green's Theorem, as noted above, is a generalization of the Fundamental Theorem of Calculus: both theorems show that the integral of the derivative of a function (in an appropriate sense) on a region can be computed using only the values of the function on the boundary of the region.
    - Remark: The hypotheses about the curve ("simple closed rectifiable, oriented counterclockwise") are to ensure the curve is nice enough for the theorem to hold. "Simple" means that the curve does not cross itself, "closed" means that its starting point is the same as its ending point (e.g., a circle), "rectifiable" means "piecewise-differentiable" (i.e., differentiable except at a finite number of points), and "oriented counterclockwise" means that C runs around the boundary of R in the counterclockwise direction.
    - It essentially suffices to prove Green's Theorem for rectangular regions, as more complicated regions can be built by "gluing together" simpler ones (in much the manner of a Riemann sum); overlapping boundary pieces on two rectangles sharing a side will have opposite orientations and will therefore cancel out.
    - Proof (rectangular regions): for a rectangular region $a \le x \le b$, $c \le y \le d$, we have $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$, where C_1, C_2, C_3, and C_4 are the four sides of the rectangle (with the proper orientation), and the function to be integrated on each curve is $P\,dx + Q\,dy$.
    - Setting up parametrizations shows $\int_{C_1} [P\,dx + Q\,dy] + \int_{C_3} [P\,dx + Q\,dy] = \int_a^b [P(x, c) - P(x, d)]\,dx$, and $\int_{C_2} [P\,dx + Q\,dy] + \int_{C_4} [P\,dx + Q\,dy] = \int_c^d [Q(b, y) - Q(a, y)]\,dy$.
    - For the double integral we have $\iint_R -\dfrac{\partial P}{\partial y}\,dy\,dx = \int_a^b \int_c^d -\dfrac{\partial P}{\partial y}\,dy\,dx = \int_a^b [P(x, c) - P(x, d)]\,dx$, and $\iint_R \dfrac{\partial Q}{\partial x}\,dx\,dy = \int_d^c \int_a^b \dfrac{\partial Q}{\partial x}\,dx\,dy = \int_c^d [Q(b, y) - Q(a, y)]\,dy$.
    - By comparing the expressions, we see that $\int_C [P\,dx + Q\,dy] = \iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dy\,dx$, as desired.

- Green's Theorem can be used to convert line integrals into double integrals, which can often be easier to evaluate if the curve is complicated but the region it encloses is simpler to describe.

- Example: Evaluate the integral $\oint_C 3x^2\,dx + 2xy\,dy$, where C is the counterclockwise boundary of the triangle having vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$.
    - We will evaluate the integral both as a line integral and using Green's Theorem.
    - Green's Theorem says that $\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dy\,dx$, so setting $P = 3x^2$ and $Q = 2xy$ produces $\oint_C 3x^2\,dx + 2xy\,dy = \iint_R 2y\,dy\,dx$, where R is the interior of the triangle.
    - To compute the double integral, we need to describe the region R. A quick sketch shows that R is defined by the inequalities $0 \le x \le 1$ and $0 \le y \le 2 - 2x$.
    - Thus, the double integral is $\int_0^1 \int_0^{2-2x} 2y\,dy\,dx = \int_0^1 (y^2)\Big|_{y=0}^{2-2x}\,dx = \int_0^1 (2 - 2x)^2\,dx = \boxed{\dfrac{4}{3}}$.
    - To compute the line integral, we need to parametrize each piece of the boundary. There are three pieces.
        1. The segment from $(0, 0)$ to $(1, 0)$, parametrized by $x = t$, $y = 0$ for $0 \le t \le 1$. Then $dx = dt$ and $dy = 0$, so the integral here is $\int_0^1 3t^2\,dt = 1$.

2. The segment from $(1,0)$ to $(1,2)$, parametrized by $x=1$, $y=t$ for $0 \le t \le 2$. Then $dx=0$ and $dy=dt$, so the integral here is $\int_0^2 2t\,dt = 4$.
3. The segment from $(1,2)$ to $(0,0)$, parametrized by $x=1-t$, $y=2-2t$ for $0 \le t \le 1$. Then $dx=-dt$ and $dy=-2dt$, so the integral here is $\int_0^1 \left[3(1-t)^2\cdot(-dt)+2(1-t)(2-2t)\cdot(-2dt)\right] = \int_0^1 [-11t+22t-11t^2]\,dt = -\dfrac{11}{3}$.

  - Thus, the value of the line integral over the entire boundary is the sum of these three, namely $1+4-\dfrac{11}{3} = \boxed{\dfrac{4}{3}}$.
  - As dictated by Green's theorem, we get the same result either way. However, the double integral was quite a bit less work!

- We can use Green's Theorem to simplify the calculation of circulation and flux integrals on closed curves.
  - Specifically, we can use the theorem to give expressions for circulation and flux either as line integrals or as double integrals over a region.
  - Depending on the shape of the region and its boundary, and the nature of the field $\mathbf{F}$, either the line integral or the double integral can be easier.

- Theorem (Green's Theorem, Tangential Form): If C is a simple closed rectifiable curve oriented counterclockwise and R is the region it encloses, and $\mathbf{F}$ is a continuously differentiable vector field, then the circulation around C is equal to $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\,dA$.
  - Recall that if $\mathbf{F}=\langle P,Q\rangle$, then $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \left\langle 0,0,\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}\right\rangle$ and $(\operatorname{curl}\mathbf{F})\cdot\mathbf{k} = \dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}$. The curl measures how much the vector field is rotating around a given point.
  - Thus, if we write everything out in terms of vector field components, the tangential form of Green's Theorem reads $\oint_C P\,dx+Q\,dy = \iint_R \left(\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}\right)dy\,dx$, which is just the statement we gave above.

- Theorem (Green's Theorem, Normal Form): If C is a simple closed rectifiable curve oriented counterclockwise and R is the region it encloses, and $\mathbf{F}$ is a continuously differentiable vector field, then the flux across C is equal to $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \iint_R (\operatorname{div}\mathbf{F})\,dA$.
  - Here, if $\mathbf{F}=\langle P,Q\rangle$ then $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial P}{\partial x}+\dfrac{\partial Q}{\partial y}$. This is called the divergence of $\mathbf{F}$ and measures how much the vector field is pushing inward or outward at the given point.
  - Explicitly, the normal form of Green's Theorem reads $\oint_C P\,dy - Q\,dx = \iint_R \left(\dfrac{\partial P}{\partial x}+\dfrac{\partial Q}{\partial y}\right)dy\,dx$, which we can recognize as the original statement of Green's Theorem except with $-Q$ in place of P and P in place of Q.
  - There is a nice interpretation of the normal form of Green's Theorem: imagine that $\mathbf{F}$ is modeling population movement, and that C is the border of a country taking up the region R. At a city along the border C, the value $\mathbf{F}\cdot\mathbf{N}$ measures the immigration (in or out) to that city from across the border. At a city inside the country, the value $\operatorname{div}\mathbf{F}$ measures the net immigration (into or out of) that city.
  - The normal form of Green's Theorem then says: if we add up the net immigration along the border, this equals the total population flow inside the country. (These two quantities are definitely equal, since they both tally the net immigration into the country as a whole.)

- Example: Find the outward flux through, and the (counterclockwise) circulation around, the square with vertices $(0,0)$, $(2,0)$, $(2,2)$, and $(0,2)$, for the vector field $\mathbf{F}(x,y) = \langle x^2-2xy,\, y^3-x\rangle$.

- We could parametrize the boundary of this region and evaluate the line integrals to find the flux and circulation. However, this would be very tedious, as it requires computing four line integrals each time (one for each side of the square). We can save a lot of effort by using Green's Theorem, which applies because the boundary is a closed curve.
- For the flux, Green's Theorem says that Flux across $C = \oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dy\, dx$.
- Here, we have $P = x^2 - 2xy$ and $Q = y^3 - x$, and the region is $0 \le x \le 2$ and $0 \le y \le 2$.
- Therefore, since $\frac{\partial P}{\partial x} = 2x - 2y$ and $\frac{\partial Q}{\partial y} = 3y^2$, the flux is
$\int_0^2 \int_0^2 \left(2x - 2y + 3y^2\right) dy\, dx = \int_0^2 \left(2xy - y^2 + y^3\right)\Big|_{y=0}^{2} dx = \int_0^2 (4x + 4)\, dx = \boxed{16}$.
- Green's Theorem also says that Circulation around $C = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\, dx$.
- Since $\frac{\partial Q}{\partial x} = -1$ and $\frac{\partial P}{\partial y} = -2x$, the circulation is $\int_0^2 \int_0^2 (-1 + 2x)\, dy\, dx = \int_0^2 (-2 + 4x)\, dx = \boxed{4}$.

- Example: For $\mathbf{F}(x, y) = \langle -x^2 y,\, xy^2 \rangle$, find the outward flux through and the (counterclockwise) circulation around the circle $x^2 + y^2 = 4$.
  - We apply Green's Theorem: in this case, the region R is the region $x^2 + y^2 \le 4$.
  - The flux is $\iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dy\, dx = \iint_R (-2xy + 2xy)\, dA = \iint_R 0\, dA = \boxed{0}$.
  - The circulation is $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\, dx = \iint_R (y^2 + x^2)\, dA = \int_0^{2\pi} \int_0^2 r^2 \cdot r\, dr\, d\theta = \boxed{8\pi}$, upon switching to polar coordinates.

- One of the many applications of Green's Theorem is to give various ways to compute the area of a planar region using a line integral around its boundary. Specifically, if C is the counterclockwise boundary curve of the region R (and C and R satisfy the hypotheses of Green's Theorem), then

$$\text{Area of } R = \oint_C x\, dy = \oint_C -y\, dx = \oint_C \frac{1}{2}(x\, dy - y\, dx)$$

because by Green's Theorem, each of the line integrals is equal to $\iint_R 1\, dy\, dx$, which is the area of R.
  - One physical application of this idea is the construction of planimeters: they are devices used for measuring the area of a region that operate by tracing along its boundary.
  - The basic principle is that the planimeter measures the amount of movement perpendicular to its measuring arm: integrating the resulting dot product around the boundary of the curve, per Green's theorem, then yields the area.

- Example: Compute the area enclosed by the ellipse $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$.
  - Using the third formula, we compute
$A = \oint_C \frac{1}{2}(x\, dy - y\, dx) = \int_0^{2\pi} \frac{1}{2}\left[(a\cos t)(b\cos t) - (b\sin t)(-a\sin t)\right] dt = \int_0^{2\pi} \frac{ab}{2}\, dt = \boxed{\pi ab}$.

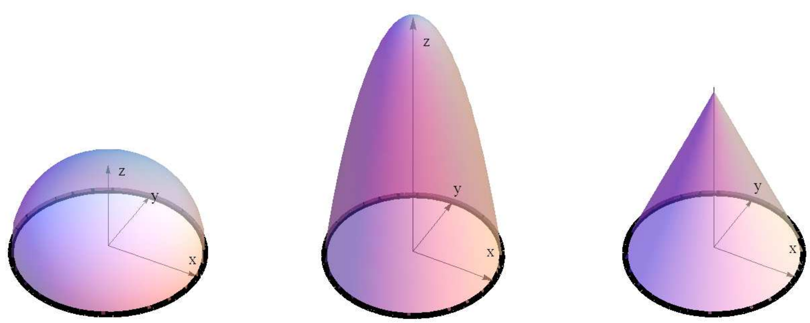
4.6 Stokes's Theorem and Gauss's Divergence Theorem

- We now discuss two generalizations of Green's theorem to 3 dimensions: these are Stokes's Theorem and Gauss's Divergence Theorem.
  - As with Green's Theorem, these theorems can be used in either direction, depending on which integral is easier to set up and evaluate.

* Indeed, taken together, the Fundamental Theorem of Calculus for line integrals, Green's Theorem, Stokes's Theorem, and Gauss's Divergence Theorem collectively unify all of our notions of integration, and are all different generalizations of the Fundamental Theorem of Calculus.
* They all relate the integral of a function on the boundary of a region to the integral of a derivative on the interior of the region.
* Symbolically, their statements all read as $\int_{\partial R} d\omega = \int_{R} \omega$, where $d\omega$ represents an appropriate differential of a function ω and ∂R represents the boundary of the region R.

4.6.1 Stokes's Theorem

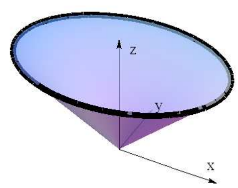
- We begin with Stokes's theorem, which is the 3-dimensional version of the tangential form of Green's theorem:
- Theorem (Stokes's Theorem): If C is a simple closed rectifiable curve in 3-space that is oriented counterclockwise around the surface S, and $\mathbf{F}$ is a continuously differentiable vector field, then the circulation around C is given by $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{T}$ is the unit tangent to the curve and $\mathbf{n}$ is the unit normal to the surface.
  * Important Note: The curve C must run counterclockwise around S: in other words, when walking along C, the surface should be on its left-hand side. If one wishes to set up a problem where a curve runs clockwise around a surface, it is equivalent to traversing the curve in the opposite direction, and so the integral will be scaled by -1.
  * The hypotheses about the curve ("simple closed rectifiable, oriented counterclockwise") are the same as in Green's Theorem, and they ensure the curve is nice enough for the theorem to hold.
  * Recall $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$ if $\mathbf{F} = \langle P, Q, R \rangle$.
  * Intuitively, if we think of a vector field as modeling the flow of a fluid, the quantity $(\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}$ at (x, y, z) measures how much the fluid is circulating around the point (x, y, z) along the surface. Stokes's Theorem then says: we can measure how much the fluid circulates around the whole surface by measuring how much it circles around its boundary.
  * The proof of Stokes's Theorem (which we omit) is essentially the same as the proof of Green's Theorem: we can reduce to the case of showing the result for "simple" patches on the surface. Then, by parametrizing the patches explicitly, we can show Stokes's Theorem is essentially the same as the tangential form of Green's Theorem on each patch.
- Stokes's Theorem generalizes the tangential form of Green's Theorem to cover 3-dimensional closed curves and the surfaces they bound. Note that unlike in Green's Theorem, there are many possible surfaces that any given curve can bound.
  * For example, the unit circle $x^2 + y^2 = 1$, $z = 0$ in the xy-plane bounds the upper portions (i.e., where $z \geq 0$) of the sphere $x^2 + y^2 + z^2 = 1$, the paraboloid $z = 2(1 - x^2 - y^2)$, and the cone $z = 1 - \sqrt{x^2 + y^2}$, as pictured below:



- Typically, we use Stokes's Theorem when the line integral over the boundary is difficult, but there is a nicer surface available.

- Example: Find the circulation of the field $\mathbf{F}(x,y,z) = \langle y^2z^3,\, 2xyz^3,\, 3xy^2z^2\rangle$ around the ellipse given by the intersection of the upper half of the ellipsoid $x^2+2y^2+2z^2=12$ with the cone $x^2+2y^2=z^2$.

  - Here is a picture of the surfaces and the ellipse:



  - We could write down a parametrization for this ellipse with a little bit of effort: substituting the cone's equation into the sphere's equation gives $3z^2 = 12$ hence $z = 2$. Then using the fact that $x^2+2y^2=4$ is parametrized by $x = 2\cos(t)$ and $y = \sqrt{2}\sin(t)$ gives us a parametrization for the curve as $\mathbf{r}(t) = \langle 2\cos(t), \sqrt{2}\sin(t), 2\rangle$. The resulting circulation integral does not look so wonderful, although it is possible to evaluate it.
  - Another way is to try to use Stokes's Theorem. We have two obvious surfaces to choose from (ellipsoid and cone); since the curve runs counterclockwise around the ellipsoid, we will use that.
  - Stokes's Theorem tells us that Circulation around $C = \oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,d\sigma$.
  - We have $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \langle 6xyz^2 - 6xyz^2, 3y^2z^2 - 3y^2z^2, 2yz^3 - 2yz^3\rangle = \langle 0,0,0\rangle$.
  - So the curl of $\mathbf{F}$ is zero. Hence $(\operatorname{curl}\mathbf{F})\cdot\mathbf{n}$ will also be zero, so we see that the circulation is $\boxed{0}$, without even having to set up the surface integral.

- Example: Find the flux of the curl $\iint_S \operatorname{curl}(\mathbf{F})\cdot\mathbf{n}\,d\sigma$, where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + e^{x+y}\mathbf{k}$, S is the surface which is the part of the sphere $x^2+y^2+z^2=25$ below the plane $z=3$, and $\mathbf{n}$ is the outward normal.

  - We will use Stokes's Theorem. In this case, we want S to be the part of the sphere $x^2+y^2+z^2=25$ which is below the plane $z=3$.
  - The boundary of this surface will be the intersection of the plane and the sphere: we see that the curve is the set of points $\{(x,y,z): x^2+y^2=16,\ z=3\}$, which is a circle that we can parametrize as $\mathbf{r}(t) = \langle 4\cos(t),\, 4\sin(t),\, 3\rangle$ for $0 \le t \le 2\pi$.
  - However: the surface S lies below the curve C, not above it: so, when viewed from below (which is required because we are using the the outward normal), the curve runs clockwise around the surface.
  - In order to apply Stokes's Theorem, we need to reverse the orientation of the curve C, which we can do by interchanging the limits of integration: thus we start at $t=2\pi$ and end at $t=0$.
  - From Stokes's Theorem, the flux of the curl is given by the line integral $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$.
  - We have $P = 12\sin(t)$, $Q = -12\cos(t)$, and $R = e^{4\cos(t)+4\sin(t)}$, and also $dx = -4\sin(t)\,dt$, $dy = 4\cos(t)\,dt$, and $dz = 0\,dt$.
  - We get $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{2\pi}^{0}\left[(12\sin(t))\cdot(-4\sin(t)\,dt) + (-12\cos(t))\cdot(4\cos(t)\,dt)) + e^{4\cos(t)+4\sin(t)}\cdot 0\,dt\right] = \int_{2\pi}^{0} -48\,dt = \boxed{96\pi}$.

4.6.2 Gauss's Divergence Theorem

- Now we discuss Gauss's Divergence Theorem, which is the 3-dimensional version of the normal form of Green's theorem:
- Theorem (Gauss's Divergence Theorem): If S is a closed, bounded, piecewise-smooth surface that fully encloses a solid region D, and $\mathbf{F}$ is a continuously differentiable vector field, then the flux across S is given by $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D (\text{div } \mathbf{F})\, dV$, where $\mathbf{n}$ is the outward unit normal to the surface.
  - To get an idea of the setup, if S is the unit sphere $x^2 + y^2 + z^2 = 1$, then D would be the unit ball $x^2 + y^2 + z^2 \le 1$. If S consists of the 6 faces of the unit cube, then D would be the interior of the cube.
  - Here, if $\mathbf{F} = \langle P, Q, R \rangle$ then div $\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$.
  - Intuitively, if we think of a vector field as modeling the flow of a fluid, the divergence measures whether there is a "source" or a "sink" at a given point (i.e., whether fluid is flowing inward toward that point, or outward from that point). The flux through a surface measures how much fluid is flowing across the surface.
  - The Divergence Theorem then says that we can measure how much fluid is flowing in or out of a solid region by measuring how much fluid is flowing across its boundary.
  - The proof of the Divergence Theorem (which we omit) is essentially the same as the proof of Green's Theorem: we reduce to the case of showing the result for rectangular boxes, and then parametrize the boxes explicitly.
- Typically, we want to use the Divergence Theorem to compute the flux through a closed surface, since it is usually easier to evaluate the triple integral than the surface integral.
- Example: Find the outward flux of the field $\mathbf{F}(x, y, z) = \langle x^3 - 3y, 2yz + 1, xyz \rangle$ through the cube bounded by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$.
  - We could do this directly by computing the flux across each of the six faces of the cube. This is not the best idea, because it would require setting up six surface integrals.
  - Instead, we use the Divergence Theorem: it says Flux across $S = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_V (\text{div } \mathbf{F})\, dV$.
  - The solid region V is defined by $-1 \le x \le 1$, $-1 \le y \le 1$, $-1 \le z \le 1$, and div $\mathbf{F} = (3x^2) + (2z) + (xy)$.
  - Thus, the flux integral is

$$\begin{aligned} \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \left(3x^2 + 2z + xy\right) dz\, dy\, dx &= \int_{-1}^{1}\int_{-1}^{1} \left(3x^2 z + z^2 + xyz\right)\Big|_{z=-1}^{1} dy\, dx \\ &= \int_{-1}^{1}\int_{-1}^{1} \left(6x^2 + 2xy\right) dy\, dx \\ &= \int_{-1}^{1} \left(6x^2 y + xy^2\right)\Big|_{y=-1}^{1} dx = \int_{-1}^{1} 12x^2\, dx = \boxed{8}. \end{aligned}$$

- Example: Compute the flux $\oiint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = (x^3 + yz)\mathbf{i} + (y^3 + xz)\mathbf{j} + (z^3 + xy)\mathbf{k}$, S is the unit sphere $x^2 + y^2 + z^2 = 1$, and $\mathbf{n}$ is the outward normal.
  - We will use the Divergence Theorem. If $\mathbf{F} = \langle P, Q, R \rangle$ then $\text{div}(\mathbf{F}) = P_x + Q_y + R_z$, so here we have $\text{div}(\mathbf{F}) = 3x^2 + 3y^2 + 3z^2$.
  - The region enclosed by S is the unit ball $x^2 + y^2 + z^2 \le 1$.
  - Thus the triple integral is $\displaystyle\iiint_{x^2+y^2+z^2 \le 1} \left(3x^2 + 3y^2 + 3z^2\right) dz\, dy\, dx$.
  - To evaluate this integral we switch to spherical coordinates: the region is bounded by the inequalities $0 \le \rho \le 1$, $0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$, the function is $3\rho^2$, and the differential is $\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$.
  - So we obtain $\int_0^{2\pi}\int_0^{\pi}\int_0^1 3\rho^2 \cdot \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \dfrac{3}{5}\sin(\phi)\, d\phi\, d\theta = \int_0^{2\pi} \dfrac{6}{5}\, d\theta = \boxed{\dfrac{12\pi}{5}}$.

4.7 Applications of Vector Calculus

- In this section we discuss a number of moderately related applications of vector calculus. Many of our examples are drawn from physics and engineering, but we also mention some examples that show up in other fields.

4.7.1 Newton's Law of Gravitation and Kepler's Laws

- We begin by discussing planetary motion using some of our properties of vectors and vector fields. The main tool we will use for our analysis is Newton's law of gravitation, which says that the gravitational attraction imparted by an object on a particle is directly proportional to each of their mass and inversely proportional to the square of the distance between them.
  - The simplest situation is with an object and a particle. Explicitly, suppose the particle has mass m at $\mathbf{r} = \langle x, y, z\rangle$, and the object has mass M and is located at the origin $(0, 0, 0)$.
  - The unit vector from $\mathbf{r}$ to the origin is $-\dfrac{\mathbf{r}}{||\mathbf{r}||}$, and the magnitude of the field $\mathbf{F}$ is equal to a constant times m times M times $\dfrac{1}{||\mathbf{r}||^2}$. The constant of proportionality here is called G, the universal gravitational constant, and its value has been measured to be $6.674 \mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)$.
  - Putting all of this together shows that $\mathbf{F} = \dfrac{GmM}{||\mathbf{r}||^2} \cdot \left(-\dfrac{\mathbf{r}}{||\mathbf{r}||}\right) = -\dfrac{GmM}{||\mathbf{r}||^3}\mathbf{r}$, which we can explicitly write as a function of x, y, z: $\mathbf{F}(x, y, z) = -\dfrac{GmM}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z\rangle$.
  - If the object is located at a different point, we can simply translate the underlying coordinate system: explicitly, if the object is instead located at (a, b, c), the gravitational field would be $\mathbf{F}(x, y, z) = -\dfrac{GmM}{((x-a)^2 + (y-b)^2 + (z-c)^2)^{3/2}} \langle x-a, y-b, z-c\rangle$.
  - If we have several objects, we then sum all of their contributions to the graviational field. For example, if we had an object of mass M_1 at $(0, 0, 0)$ and another object of mass M_2 at $(2, 0, 1)$, then the resulting field would be $\mathbf{F}(x, y, z) = -\dfrac{GmM_1}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z\rangle - \dfrac{GmM}{((x-2)^2 + y^2 + (z-1)^2)^{3/2}} \langle x-2, y, z-1\rangle$.
- Once we have written down the gravitational field, we can use Newton's second law ($\mathbf{F} = m\mathbf{a}$) to write down equations of motion for objects passing through the gravitational field.
  - Explicitly, suppose that the path $\mathbf{r}$ describes the motion of a particle (a planet of mass m) through space, and the only force acting on the planet is the gravity of the sun (mass M) at the origin.
  - Then the gravitational force is $\mathbf{F}(\mathbf{r}) = -\dfrac{GmM}{||\mathbf{r}||^3}\mathbf{r}$, and so by Newton's second law, we obtain the equation $\mathbf{a} = -\dfrac{GM}{||\mathbf{r}||^3}\mathbf{r}$. Since $\mathbf{a} = \mathbf{r}''$, we obtain the equation $\mathbf{r}''(t) = -\dfrac{GM}{||\mathbf{r}(t)||^3}\mathbf{r}(t)$.
  - This is a second-order differential equation for $\mathbf{r}$ which (in general) is quite difficult to solve explicitly even in simple cases. We could write it out explicitly in terms of the components of $\mathbf{r}$, which would yield a system of three second-order differential equations for the components x, y, and z, but that does not really make the system any easier to solve.
  - Rather than attempting to solve the system, we will make a few observations about the solutions.
- First, we claim that the planet's orbit lies in a plane passing through the sun.
  - To see this, observe that our equation $\mathbf{a} = -\dfrac{GM}{||\mathbf{r}||^3}\mathbf{r}$ tells us that $\mathbf{a}$ is a scalar multiple of $\mathbf{r}$.
  - Now consider the vector $\mathbf{n} = \mathbf{r} \times \mathbf{v}$. Using the product rule for cross products, we see that the derivative of this vector is $\dfrac{d}{dt}[\mathbf{r} \times \mathbf{v}] = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a}$. But both terms in this sum are zero, because the cross product of any vector with itself is zero, and $\mathbf{a}$ is a scalar multiple of $\mathbf{r}$.

* Thus, $\frac{d}{dt}[\mathbf{r} \times \mathbf{v}]$ is zero, and so $\mathbf{n} = \mathbf{r} \times \mathbf{v}$ is a constant vector.
* But this means that the position and velocity of the particle both lie in the plane passing through its starting position whose normal vector is $\mathbf{n}$, and by our relation, all of the higher derivatives of the particle's position will also lie in this plane. This means that the particle's motion will stay in the plane as time moves forward.

- By extending this sort of analysis, one may derive Kepler's famous laws of planetary motion:

  1. The orbit of a planet is a conic section with the sun at one focus. Specifically, the conic's eccentricity is $e = \frac{r_0 v_0^2}{GM} - 1$.
  2. The radius vector $\mathbf{r}$ from the sun to the planet sweeps out equal areas in equal times.
  3. The square of the orbital period T is proportional to the cube of the length of the semimajor axis a. Specifically, $\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$.

- The best approach to establishing Kepler's laws is to work in polar coordinates. This approach is feasible because, as we have just shown, the orbit of a planet lies in a plane passing through the sun.

  * For notational convenience we will use dots to denote time-derivatives.
  * Place the sun at the origin and define $r = ||\mathbf{r}||$ to be the radial parameter, with angle parameter θ, and also define the unit vector $\mathbf{u}_r = \langle \cos\theta, \sin\theta, 0 \rangle$ in the direction from the sun to the planet and its orthogonal unit vector $\mathbf{u}_\theta = \langle -\sin\theta, \cos\theta, 0 \rangle$, which is the unit vector in the direction of increasing θ.
  * We see $\frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta$ and $\frac{d\mathbf{u}_\theta}{d\theta} = -\mathbf{u}_r$, so the chain rule gives $\dot{\mathbf{u}}_r = \frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\mathbf{u}_\theta$ and $\dot{\mathbf{u}}_\theta = \frac{d\mathbf{u}_\theta}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\mathbf{u}_r$.
  * Then because $\mathbf{r} = r\mathbf{u}_r$, by the product rule we get $\mathbf{v} = \dot{\mathbf{r}} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{dt} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta$.
  * We can then derive the second law by calculating the vector $\mathbf{n} = \mathbf{r} \times \mathbf{v}$ from earlier: explicitly, we have $\mathbf{n} = (r\mathbf{u}_r) \times (\dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta) = r\dot{r}(\mathbf{u}_r \times \mathbf{u}_r) + r^2\dot{\theta}(\mathbf{u}_r \times \mathbf{u}_\theta) = \left\langle 0, 0, r^2\dot{\theta} \right\rangle$.
  * Now we can compute the area swept out by the radius vector between time $t = t_1$ and time $t = t_2$. Using integration in polar coordinates and then a substitution in the resulting line integral, this is $\int_{\theta_1(t)}^{\theta_2(t)} \int_0^{r(t)} 1 \cdot r\, dr\, d\theta = \int_{\theta_1(t)}^{\theta_2(t)} \frac{1}{2}r^2\, d\theta = \int_{t_1}^{t_2} \frac{1}{2}r(t)^2\dot{\theta}(t)\, dt$.
  * However, the integrand in the last integral is exactly the z-component of the constant vector $\mathbf{n}$, so the integral is simply $(t_2 - t_1)$ times a constant. This means the area depends only on the amount of time $t_2 - t_1$, which (when phrased more elegantly) is Kepler's second law above.
  * The other laws are a bit more difficult and require careful manipulation of the differential equation $\mathbf{r}''(t) = -\frac{GmM}{||\mathbf{r}||^3}\mathbf{r}$.
  * We will not go through the details of these other than to say that the first law boils down to computing the polar equation for r in terms of θ, and verifying it has the form $r = \frac{(1+e)r_0}{1 + e\cos\theta}$ where r_0 is the radius at perihelion (i.e., the minimal radius), which we take to occur at time $t = 0$ and $\theta = 0$, while the third law boils down to comparing two formulas for the area of an ellipse (one of them is the integral formula above, and the other is π times the semimajor axis times the semiminor axis, which can be derived using a change of variables in a double integral).

- We can also calculate the work done by a gravitational field $\mathbf{F}(\mathbf{r}) = -\frac{GmM}{||\mathbf{r}||^3}\mathbf{r} = -\frac{GmM}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z \rangle$.

  * Calculating the work integral as a line integral directly is quite messy because of the square root factor in the denominator.
  * But, as our physical intuition would suggest, and as we can check explicitly by computing $\operatorname{curl}\mathbf{F} = \langle 0, 0, 0 \rangle$, the gravitational field $\mathbf{F}$ is conservative.
  * If we search for a potential function, we can eventually see that $U = GmM(x^2+y^2+z^2)^{-1/2}$ has the property that $\mathbf{F} = \langle U_x, U_y, U_z \rangle$, as the chain rule terms $2x$, $2y$, $2z$ exactly give the needed factors of x, y, z in the three components.

- We can therefore easily compute the work done by the vector field on a particle that travels from point $\mathbf{a}$ to point $\mathbf{b}$: specifically, by the fundamental theorem of line integrals, the work done by $\mathbf{F}$ is equal to $U(\mathbf{b}) - U(\mathbf{a}) = \dfrac{GmM}{||\mathbf{b}||} - \dfrac{GmM}{||\mathbf{a}||}$.
- Notice that the work only depends on the distances of the points from the origin, so the work done by gravity will be the same regardless of the relative positions of the start and end points on the spheres of constant radius centered at the origin.

- We can even apply this formula to compute the approximate graviational potential on the surface of the earth.
  - If the start and end points are both approximately a distance R from the origin, then we can estimate the change in the potential energy using a linearization (or, equivalently, a directional derivative).
  - It is straightforward to compute that the linearization of $U(x) = \dfrac{GmM}{x}$ at $x = R$ is $L(x) = \dfrac{GmM}{R} - \dfrac{GmM}{R^2}(x - R)$.
  - Therefore, the approximate value of $U(R + \Delta h) - U(R)$ is $-\dfrac{GmM}{R^2}\Delta h$, which equals $mg\Delta h$ where m is the mass of the particle, h is the change in height, and $g = \dfrac{GM}{R^2}$ is a constant.
  - If we evaluate this constant g using the known values $G = 6.674\text{m}^3/(\text{kg}\cdot\text{s}^2)$, the mass of the Earth $M = 5.972 \cdot 10^{24}\text{kg}$, and the radius of the Earth $R = 6.371 \cdot 10^6\text{m}$, we do in fact obtain the local gravitational constant $g = 9.817\text{m/s}^2$.
  - Of course, this should not be very surprising, because by Newton's second law, the magnitude of the acceleration due to the gravitational field will be $||\mathbf{F}(\mathbf{r})||/m = -GM/||\mathbf{r}||^2$.

4.7.2 The Heat and Wave Equations

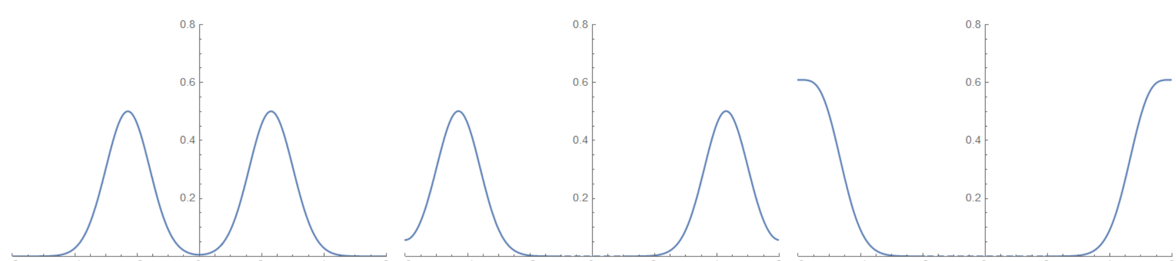
- We can model a great deal of physical phenomena using fairly simple models, and we can use vector calculus to study the behaviors of the models quite fruitfully in many cases.
  - Here, we will discuss two very fundamental models: the heat equation and the wave equation.
- The heat equation in its general form is the partial differential equation $f_t = \gamma\nabla \cdot (\nabla f)$.
  - For a function $f(x, y, z, t)$, the heat equation reads as $f_t = \gamma(f_{xx} + f_{yy} + f_{zz})$ in standard notation.
  - The function $f(x, y, z, t)$ models the temperature of an object at a point (x, y, z) at time t, and γ is a rate constant.
  - For shorthand (even though it is technically bad notation) we often write the operator $\nabla \cdot \nabla$ as $\nabla^2 = \langle \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2} \rangle$. This operator is called the Laplacian and is often also written as Δ.
  - The heat equation is a rephrasing of the second law of thermodynamics and Newton's law of cooling (heat flows from hot things to cold ones at a rate proportional to the difference in temperatures).
- In fact, we can derive the heat equation $f_t = \gamma\nabla^2 f$ from these two principles using the divergence theorem.
  - Explicitly, let $H(t)$ be the amount of heat contained in a region D. Then $H(t) = \iiint_D \alpha f(x, y, z, t)\, dV$ since temperature is a measure of heat density.
  - Taking the derivative with respect to t of both sides yields $H_t(t) = \dfrac{d}{dt}\left[\iiint_D \alpha f(x, y, z, t)\, dV\right] = \iiint_D \alpha f_t(x, y, z, t)\, dV$, where the second equality follows from a general theorem of Leibniz on interchanging the order of a derivative and an integral (it is, essentially, a combination of the mixed-partials theorem and Fubini's theorem, and it allows us to move the t-derivative inside the integral).
  - Since heat can only flow into the solid region D across its boundary, the heat flow $H_t(t)$ is also given by computing the flux of the heat flowing through the boundary of the surface.

- The vector field modeling the heat flow is ∇f, so the flux of this field is the surface integral $\iint_S \beta(\nabla f)\cdot \mathbf{n}\,d\sigma$.
- By the divergence theorem, the surface integral is also equal to $\iiint_D \beta\nabla\cdot(\nabla f)\,dV = \iiint_D \beta\nabla^2 f\,dV$.
- By comparing to the triple-integral expression from earlier, we obtain an equality $\iiint_D \alpha f_t\,dV = \iiint_D \beta\nabla^2 f\,dV$ on every solid D. The only way this can happen is if the underlying functions αf_t and $\beta\nabla^2 f$ are equal everywhere.
- Moving the constant factors around to solve for f_t immediately yields the heat equation $f_t = \gamma\nabla^2 f$.

- The heat equation $f_t = \gamma\nabla^2 f$, as an abstract differential equation, also shows up in various other places.
  - In probability theory, the heat equation shows up as a very natural continuous model for random walks. In physics, this is closely connected with the study of Brownian motion.
  - In financial mathematics, the Black-Scholes equation (which is used for computing the proper price of options) is a minor variation of the heat equation: if V is the price of an option as a function of the asset S and time t, then it says $V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$. Up to the coefficients and the first-order and constant terms, it is essentially $V_t = \gamma V_{SS}$, which is a one-dimensional heat equation.
  - In quantum mechanics, Schrödinger's equation reads as $H|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$. For a single particle, H is (essentially) the Laplacian operator ∇^2, so this is (very roughly!) a heat equation with an imaginary constant factor.

- Our other fundamental differential equation is the <u>wave equation</u>: $f_{tt} = \gamma\nabla^2 f$.
  - For a function $f(x,y,z,t)$, the wave equation reads as $f_{tt} = \gamma(f_{xx}+f_{yy}+f_{zz})$ in standard notation.
  - Note the similarity to the heat equation: the only difference is that the wave equation has a double t-derivative rather than the single t-derivative as in the heat equation.
  - The wave equation, as one would expect from its name, models the intensity of a wave at a point (x,y,z) in space at time t.

- Pleasantly enough, the one-dimensional wave equation $f_{tt} = c^2 f_{xx}$ can actually be solved essentially explicitly.
  - The idea is to change variables and write $a = x - ct$ and $b = x + ct$.
  - Then by two applications of the multivariable chain rule, we can verify that the wave equation is equivalent to $f_{ab} = 0$, which by antidifferentiating twice can be seen to have the simple solution $f(a,b) = F(a)+G(b)$ for arbitrary functions F and G.
  - Plugging back in yields a general solution $f(x,t) = F(x-ct)+G(x+ct)$. This is the sum of a "left-moving function" $F(x-ct)$ and a "right-moving function" $G(x+ct)$ as t increases.
  - Below is plotted a solution to the one-dimensional wave equation for a fixed value of t. As t changes, the positions of the wave peaks will move at constant speed, one to the left and the other to the right:



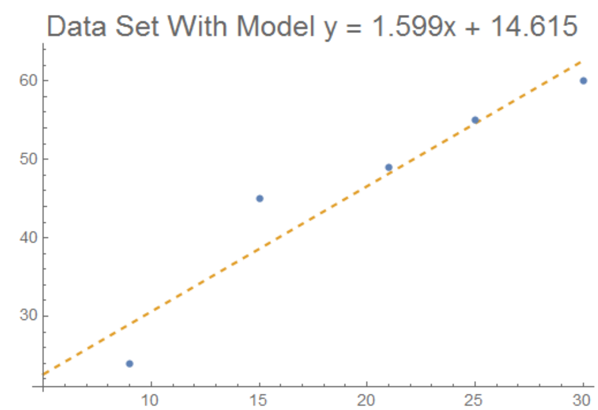
- Like the heat equation, the wave equation can also be derived from basic physical principles using the divergence theorem.
  - Specifically, suppose D is any region. Then the acceleration within D is the second t-derivative of $\iiint_D f\,dV$, which is $\iiint_D f_{tt}\,dV$.

- The vector field $\mathbf{F}$ modeling the force imparted by the wave is ∇f, and so the total force acting on D through its boundary S is equal to the surface integral $\iint_S (\nabla f) \cdot \mathbf{n} \, d\sigma$, which equals $\iiint_D \nabla^2 f \, dV$ by the divergence theorem.
- Applying Newton's second law ($F = ma$) and equating the two triple integrals on every D, using the same logic as we used earlier for the heat equation, then gives the wave equation: $f_{tt} = \gamma \nabla^2 f$.

4.7.3 Numerical Methods in Modeling Applications

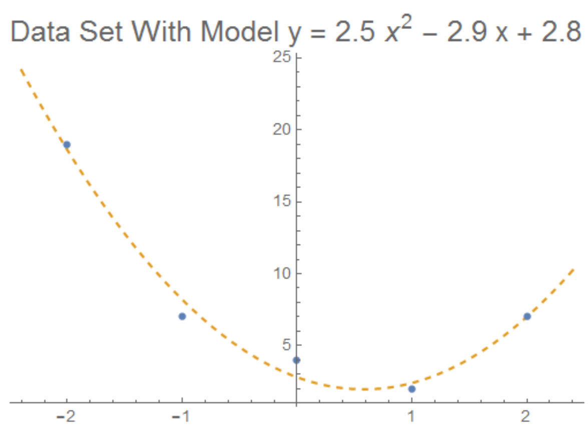
- Although we are able to write down the solutions to the one-dimensional wave equation in a convenient way, in most cases, the differential equation (or equations) modeling a physical phenomenon are difficult if not impossible to solve exactly. As such, we often want to find methods of generating good approximate solutions.
  - One approach is to employ a "step method" and linearization: we take a linearization of the system and then move a small step forward in time (the idea being that for a small step, the linearization is a good approximation of the original).
  - We then iterate this procedure with the new system that has been moved forward: we linearize and then move a small step forward in time, repeatedly.
  - Techniques like this one can be used to analyze models for weather and climate, urban planning, epidemiology (e.g., during global pandemics), ecology, experimental biology, chemistry, and physics, and just about everywhere else....
- As part of these numerical methods, one often needs to search for a minimum or maximum value of some function.
  - For example, if one wants to model a chemical reaction computationally (which is now possible to do with modern supercomputers), one needs to compute minimum-energy configurations of molecules.
  - To perform such simulations, the computer must use step methods to iterate each interaction of particles in small time intervals, and search for the minimum-energy state.
  - To find such a state, one may use a "gradient-step method": compute the current energy, and then step in the opposite direction of the gradient of this energy function.
  - As we have discussed, the gradient points in the direction of maximum increase of a function, so at each stage, the search will move in the direction that lowers total energy.
  - Eventually, a gradient-step algorithm will reach a state in which the gradient is zero, which is a critical point of the energy function.
  - To determine whether the energy is actually minimized then requires classifying the resulting critical point as a local minimum, local maximum, or saddle point.
  - Of course, in actual practice, the search space is much larger than the 2-dimensional examples we have discussed (typically it has hundreds or thousands of variables).
  - But the general principle, that one may classify the type of critical point by using a higher-dimensional version of the second derivatives test, turns out to be very similar.
- In many other applications, we have a model that we want to fit to a given data set.
  - In statistics there are various methods for making "parameter estimates" of this type: indeed, a major component of statistics is about developing methods for making parameter estimates from data seta.
- A computationally convenient technique, frequently used in practice, is to employ a least-squares regression: we minimize the sum of the squared errors between the predicted and observed values.
  - The reason to use the sum of squares, rather than something else like the sum of the absolute errors, is because we can minimize the resulting function using calculus.
  - The simplest example of this kind is to fit a linear function $y = ax+b$ to a data set $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$.
  - The function to minimize for the linear model above is $E(a, b) = (ax_1 + b - y_1)^2 + (ax_2 + b - y_2)^2 + \cdots + (ax_n + b - y_n)^2$.

- To minimize this function we set the two partial derivatives $\partial E/\partial a$ and $\partial E/\partial b$ equal to zero.
- We have $\partial E/\partial a = 2x_1(ax_1 + b - y_1) + \cdots + 2x_n(ax_n + b - y_n)$ and $\partial E/\partial b = 2(ax_1 + b - y_1) + \cdots + 2(ax_n + b - y_n)$, so that $a \sum x_i^2 + b \sum x_i = \sum x_i y_i$ and $a \sum x_i + nb = \sum y_i$.
- This is a linear system for a and b, with solution $a = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$ and $b = \frac{1}{n}(\sum y_i - a \sum x_i)$.
- These two values a and b together give the equation for the famous least-squares regression line to a data set.
- Here, for example, is the plot of a data set $\{(9, 24), (15, 45), (21, 49), (25, 55), (30, 60)\}$ along with its least-squares regression line $y = 1.599x + 14.615$:



• However, the method of least squares is quite robust and can be used with many different kinds of models.

- For example, we can use more complicated functions, such as a quadratic function $y = ax^2 + bx + c$.
- The procedure is essentially the same as before: we write down the sum of squared errors and then minimize it using calculus, by setting all of the partial derivatives equal to zero.
- Here, the function is $E(a, b, c) = (ax_1^2 + bx_1 + c - y_1)^2 + \cdots + (ax_n^2 + bx_n + c - y_n)^2$.
- We then calculate $\partial E/\partial a$, $\partial E/\partial b$, and $\partial E/\partial c$ and set them equal to zero. The resulting system will be linear in a, b, c and we can then solve it to compute the predicted coefficients a, b, c.
- Here, for example, is the plot of a data set $\{(-2, 19), (-1, 7), (0, 4), (1, 2), (2, 7)\}$ along with the parabola $y = -2.5x^2 - 2.9x + 2.8$ of best fit:



4.7.4 Maxwell's Equations and Electromagnetism

- We will now give a brief discussion of Maxwell's equations of electromagnetism.
  - Here, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, ρ is electric charge density, and ϵ_0 and μ_0 are constants. (We assume no current $\mathbf{J}$ here.)
- Here are Maxwell's equations:

Law	Integral Form	Differential Form
Gauss (E)	$\oiint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma = \frac{1}{\epsilon_0} \iiint_D \rho\, dV$	$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$
Gauss (M)	$\oiint_S \mathbf{B} \cdot \mathbf{n}\, d\sigma = 0$	$\nabla \cdot \mathbf{B} = 0$
Maxwell-Faraday	$\oint_C \mathbf{E} \cdot \mathbf{T}\, ds = -\frac{d}{dt}\left[\iint_\Sigma \mathbf{B} \cdot \mathbf{n}\, d\sigma\right]$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
Ampère	$\oint_C \mathbf{B} \cdot \mathbf{T}\, ds = \mu_0\epsilon_0 \frac{d}{dt}\left[\iint_\Sigma \mathbf{E} \cdot \mathbf{n}\, d\sigma\right]$	$\nabla \times \mathbf{B} = \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

  - In the two Gauss laws, S is a closed surface enclosing the solid region D, while in the other two laws, Σ is a surface with counterclockwise boundary curve C.
- Notice that each law comes in two different forms: an integral form and a differential form. We may convert between these different forms using the divergence theorem and Stokes's theorem.
  - Explicitly, in the two Gauss laws, S is a closed surface enclosing the solid region D, so if we apply the divergence theorem, we may convert the surface integral into a triple integral.
  - For the electric field law, by the divergence theorem we have $\oiint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{E})\, dV$, so the integral form is equivalent to saying $\iiint_D (\nabla \cdot \mathbf{E})\, dV = \frac{1}{\epsilon_0} \iiint_D \rho\, dV$.
  - This equality holds on every solid region D, so the integrands $\nabla \cdot \mathbf{E}$ and ρ/ϵ_0 are equal: this is the differential form. A similar argument works for the magnetic field law.
  - In the other two laws, Σ is a surface with counterclockwise boundary C, so we can apply Stokes's theorem.
  - For the Maxwell-Faraday law, by Stokes's theorem the integral $\oint_C \mathbf{E} \cdot \mathbf{T}\, ds$ equals $\iint_\Sigma (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, d\sigma$.
  - Thus, the integral form is equivalent to $\iint_\Sigma (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, d\sigma = -\frac{d}{dt} \iint_\Sigma \mathbf{B} \cdot \mathbf{n}\, d\sigma = -\iint_\Sigma \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, d\sigma$.
  - Since this holds on every surface, the two fields $\nabla \times \mathbf{E}$ and $-\frac{\partial \mathbf{B}}{\partial t}$ must be equal, giving the differential form. A similar argument yields the two versions of Ampère's law.
- We can actually derive Gauss's law for both electricity and magnetism as a consequence of more general properties of inverse-square laws.
  - Coulomb's law says that the electric force between two particles is proportional to each of their charges and inversely proportional to the square of the distance between them. (Compare this to Newton's law of gravitation, which has the same form of inverse-square law.)
  - Via the same sort of analysis we did earlier in finding the gravitational field, we can see that, for a single point charge q at the origin, the electric field $\mathbf{E}$ equals $\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{||\mathbf{r}||^3}$.
  - We can then compute the surface integral through the sphere of radius a centered at the origin directly.
  - We can see that the unit normal vector is $\mathbf{n} = \mathbf{r}/a$, and so the surface integral in spherical coordinates is $a^2 \int_0^{2\pi} \int_0^{\pi} \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{a^3} \cdot \frac{\mathbf{r}}{a}\, d\varphi d\theta = a^2 \int_0^{2\pi} \int_0^{\pi} \frac{q}{4a^2\pi\epsilon_0}\, d\varphi d\theta = \frac{q}{\epsilon_0}$, because the dot product $\mathbf{r} \cdot \mathbf{r} = a^2$ on the sphere of radius a.
  - This agrees with the triple integral of Gauss's law for the case of a single particle (the triple integral is simply q/ϵ_0).

- This may seem like a very special case of Gauss's law, but we can actually use it to get the general version. First, we need to extend Gauss's law to arbitrary surfaces.
- So suppose we have an arbitrary closed surface T containing the origin. Choose a sphere S that encloses it and take D to be the region between the two surfaces.
- Then, by the divergence theorem, we see that $\iiint_D (\nabla \cdot \mathbf{E})\, dV = \oiint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma - \oiint_T \mathbf{E} \cdot \mathbf{n}\, d\sigma$ (the minus sign is because the normal vector for T points inward).
- For $\mathbf{E}(x,y,z) = \dfrac{q}{4\pi\epsilon_0} \dfrac{\langle x, y, z\rangle\rangle}{(x^2+y^2+z^2)^{3/2}}$, we can compute explicitly that $\nabla \cdot \mathbf{E} = 0$ for $(x,y,z) \neq (0,0,0)$. Since D does not contain the origin, the triple integral is zero, and so $\oiint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma = \oiint_T \mathbf{E} \cdot \mathbf{n}\, d\sigma$.
- This means that the Gauss law result holds for a single particle and an arbitrary surface T.
- Finally, we can use the fact that Gauss's law holds for a single particle and an arbitrary surface to obtain the result for arbitrary charge distributions and arbitrary surfaces.
- The idea is simply to sum over all of the various charges, and observe that both the surface integral and the triple integral are consistent with summing over charges.
- By taking a limit of finite sums of charges, we obtain the result for arbitrary charge distributions, and so we obtain Gauss's law for electric fields.
- For the Gauss law for magnetic fields $\oiint_S \mathbf{B} \cdot \mathbf{n}\, d\sigma = 0$, the result is quite a bit simpler: the point is that there is no magnetic equivalent of charge (this would be a "magnetic monopole", of which no experimental observation has ever been made), and so the resulting triple integral of "magnetic charge" is simply zero.

- We can make a few additional observations about the constraints imposed by Maxwell's equations.
  - Both $\mathbf{E}$ and $\mathbf{B}$ have 3 components. The two Gauss's laws each impose one condition on the components, while the other two laws each impose three conditions. So we seemingly have 8 conditions on the 6 components.
  - But in fact, there are two redundant conditions, which are accounted for by the <u>div-curl identity</u>, which says $\mathrm{div}(\mathrm{curl}(\mathbf{F})) = \nabla \cdot (\nabla \times \mathbf{F}) = 0$ for any vector field $\mathbf{F}$.
  - So in fact, there are six conditions on the six components, which is (in an appropriate sense) "exactly enough" to determine them. (In fact, it is a theorem of electrodynamics that the behavior of electric and magnetic fields is completely determined by Maxwell's equations, in the sense that there are no additional "hidden" constraints.)

- As a final remark, we will observe a connection between Maxwell's equations and electromagnetic waves.
  - Suppose that the charge density q is zero everywhere: then Gauss's law for the electric field says that $\mathrm{div}(\mathbf{E}) = 0$.
  - For any vector field $\mathbf{F}$, we have the "curl-curl" identity $\mathrm{curl}(\mathrm{curl}(\mathbf{F})) = \mathrm{grad}(\mathrm{div}(\mathbf{F})) - \nabla^2 \cdot \mathbf{F}$, which is not hard to verify just by writing it out.
  - Applying this to the vector field $\mathbf{E}$ yields $\nabla \times (\nabla \times \mathbf{E}) = \mathrm{grad}(\mathrm{div}(\mathbf{E})) - \nabla^2 \cdot \mathbf{E} = -\nabla^2 \cdot \mathbf{E}$ since $\mathrm{div}(\mathbf{E}) = 0$.
  - By Maxwell's equations we have $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$, $\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$, so by the equality of mixed partials we see $\nabla \times (\nabla \times \mathbf{E}) = \nabla \times -\dfrac{\partial \mathbf{B}}{\partial t} = -\dfrac{\partial}{\partial t}[\nabla \times \mathbf{B}] = -\dfrac{\partial}{\partial t}\left[\mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}\right] = -\mu_0 \epsilon_0 \mathbf{E}_{tt}$
  - Putting all of this together with the calculation above yields $-\nabla^2 \cdot \mathbf{E} = -\mu_0 \epsilon_0 \mathbf{E}_{tt}$, which means that $\mathbf{E}$ satisfies the wave equation! (Likewise for $\mathbf{B}$.)
  - We see, therefore, that Maxwell's equations lead directly to the phenomenon of electromagnetic waves.
  - Of course, electromagnetic waves are a quite well-understood concept in the 21st century, so it is quite important to note that Maxwell published his original papers detailing these equations, and deducing some of these consequences to unify electricity and magnetism, in 1861.

Well, you're at the end of my handout. Hope it was helpful.