Calculus III (part 2): Partial Derivatives (by Evan Dummit, 2021, v. 4.00)

Contents

2 Partial Derivatives

In this chapter, we develop the theory of differentiation for functions of several variables, and discuss applications to optimization and finding local extreme points. The development of these topics is very similar to that of one-variable calculus, save for some additional complexities due to the presence of extra variables.

2.1 Limits and Partial Derivatives

- In order to talk about derivatives in the multivariable setting, we first need to establish what limits and continuity should mean. Once we have given a suitable definition of limit, we can then define the notion of a derivative.

2.1.1 Limits and Continuity

- We begin with a definition of limits for functions of several variables.
- Definition: A function $f(x,y)$ has the limit L as $(x,y) \to (a,b)$, written as $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, if, for any $\epsilon > 0$ (no matter how small) there exists a $\delta > 0$ (depending on ϵ) with the property that for all (x,y) with $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, we have that $|f(x,y) - L| < \epsilon$.

- The square root might be a little bit unsettling. But it is just a statement about "distance": saying that $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ just means that the point (x, y) is within a distance δ of the point (a, b).
- Compare this definition to the formal definition of limit for a function of one variable: A function $f(x)$ has the limit L as $x \to a$, written as $\lim_{x \to a} f(x) = L$, if, for any $\epsilon > 0$ (no matter how small) there exists a $\delta > 0$ (depending on ϵ) with the property that for all $0 < |x - a| < \delta$, we have that $|f(x) - L| < \epsilon$.
- The multivariable definition and single-variable definitions both mean the same intuitive thing: suppose you claim that the function f has a limit L, as we get close to some point P. In order to prove to me that the function really does have that limit, I challenge you by handing you some value $\epsilon > 0$, and I want you to give me some value of δ, with the property that $f(x)$ is always within ϵ of the limit value L, for all the points which are within δ of P.
- Remark: We can also define limits of functions of more than two variables. The definition is very similar in all of those cases: for example, to talk about a limit of a function $f(x, y, z)$ as $(x, y, z) \to (a, b, c)$, the only changes are to write (x, y, z) in place of (x, y), to write (a, b, c) in place of (a, b), and to write $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$ in place of $\sqrt{(x-a)^2 + (y-b)^2}$.

- Definition: A function $f(x, y)$ is continuous at a point (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y)$ is equal to the value $f(a, b)$. In other words, a function is continuous if its value equals its limit.
  - Continuous functions of several variables are just like continuous functions of a single variable: they don't have "jumps" and they do not blow up to ∞ or $-\infty$.

- Like the one-variable case, the formal definition of limit is cumbersome and generally not easy to use, even for simple functions. Here are a few simple examples, just to give some idea of the proofs:
  - Example: Show that $\lim_{(x,y)\to(a,b)} c = c$, for any constant c.
    - In this case it turns out that we can take any positive value of δ at all. Let's try $\delta = 1$.
    - Suppose we are given $\epsilon > 0$. We want to verify that $0 < \sqrt{(x-a)^2 + (y-b)^2} < 1$ will always imply that $|c - c| < \epsilon$.
    - But this is clearly true, because $|c - c| = 0$, and we assumed that $0 < \epsilon$.
    - So (as certainly ought to be true!) the limit of a constant function is that constant.
  - Example: Show that $\lim_{(x,y)\to(2,3)} x = 2$.
    - Suppose we are given $\epsilon > 0$. We want to find $\delta > 0$ so that $0 < \sqrt{(x-2)^2 + (y-3)^2} < \delta$ will always imply that $|x - 2| < \epsilon$.
    - We claim that taking $\delta = \epsilon$ will work.
    - To see that this choice works, suppose that $0 < \sqrt{(x-2)^2 + (y-3)^2} < \epsilon$, and square everything.
    - We obtain $0 < (x-2)^2 + (y-3)^2 < \epsilon^2$.
    - Now since $(y-3)^2$ is the square of the real number $y - 3$, we see in particular that $0 \le (y-3)^2$.
    - Thus we can write $(x-2)^2 \le (x-2)^2 + (y-3)^2 < \epsilon^2$, so that $(x-2)^2 < \epsilon^2$.
    - Now taking the square root shows $|x - 2| < |\epsilon|$.
    - However, since $\epsilon > 0$ this just says $|x - 2| < \epsilon$, which is exactly the desired result.
  - Remark: Using essentially the same argument, one can show that $\lim_{(x,y)\to(a,b)} x = a$, and also (by interchanging the roles of x and y) that $\lim_{(x,y)\to(a,b)} y = b$, which means that x and y are continuous functions.

- As we would hope, all of the properties of one-variable limits also hold for multiple-variable limits; even the proofs are essentially the same. Specifically, let $f(x, y)$ and $g(x, y)$ be functions satisfying $\lim_{(x,y)\to(a,b)} f(x, y) = L_f$ and $\lim_{(x,y)\to(a,b)} g(x, y) = L_g$. Then the following properties hold:
  - The addition rule: $\lim_{(x,y)\to(a,b)} [f(x, y) + g(x, y)] = L_f + L_g$.
  - The subtraction rule: $\lim_{(x,y)\to(a,b)} [f(x, y) - g(x, y)] = L_f - L_g$.

- The <u>multiplication rule</u>: $\lim_{(x,y)\to(a,b)} [f(x,y)\cdot g(x,y)] = L_f \cdot L_g$.
- The <u>division rule</u>: $\lim_{(x,y)\to(a,b)} \left[\dfrac{f(x,y)}{g(x,y)}\right] = \dfrac{L_f}{L_g}$, provided that L_g is not zero.
- The <u>exponentiation rule</u>: $\lim_{(x,y)\to(a,b)} [f(x,y)]^a = (L_f)^a$, where a is any positive real number. (It also holds when a is negative or zero, provided L_f is positive, in order for both sides to be real numbers.)
- The <u>squeeze rule</u>: If $f(x,y) \le g(x,y) \le h(x,y)$ and $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = L$ (meaning that both limits exist and are equal to L) then $\lim_{(x,y)\to(a,b)} g(x,y) = L$ as well.

- Here are some examples of limits that can be evaluated using the limit rules:
  - <u>Example</u>: Evaluate $\lim_{(x,y)\to(1,1)} \dfrac{x^2-2y}{x+y}$.
    - By the limit rules we have $\lim_{(x,y)\to(1,1)} \left[x^2-2y\right] = \lim_{(x,y)\to(1,1)} \left[x^2\right] - 2\lim_{(x,y)\to(1,1)} [y] = 1^2 - 2 = -1$ and $\lim_{(x,y)\to(1,1)} [x+y] = \lim_{(x,y)\to(1,1)} [x] + \lim_{(x,y)\to(1,1)} [y] = 1+1 = 2$.
    - Then $\lim_{(x,y)\to(1,1)} \dfrac{x^2-2y}{x+y} = \dfrac{\lim_{(x,y)\to(1,1)} \left[x^2-2y\right]}{\lim_{(x,y)\to(1,1)} [x+y]} = \dfrac{-1}{2} = \boxed{-\dfrac{1}{2}}$.
  - <u>Example</u>: Evaluate $\lim_{(x,y)\to(0,0)} \left[x\,y\,\sin\left(\dfrac{1}{x-y}\right)\right]$.
    - If we try plugging in $(x,y) = (0,0)$ then we obtain $0\cdot 0\cdot \sin\left(\dfrac{1}{0}\right)$, which is undefined: the issue is the sine term, which also prevents us from using the multiplication property.
    - Instead, we try using the squeeze rule: because sine is between -1 and $+1$, we have the inequalities $-|xy| \le x\,y\,\sin\left(\dfrac{1}{x-y}\right) \le |x\,y|$.
    - We can compute that $\lim_{(x,y)\to(0,0)} |x\,y| = \left[\lim_{(x,y)\to(0,0)} |x|\right]\cdot\left[\lim_{(x,y)\to(0,0)} |y|\right] = 0\cdot 0 = 0$.
    - Therefore, applying the squeeze rule, with $f(x,y) = -|x\,y|$, $g(x,y) = x\,y\,\sin\left(\dfrac{1}{x-y}\right)$, and $h(x,y) = |x\,y|$, yields $\lim_{(x,y)\to(0,0)} \left[x\,y\,\sin\left(\dfrac{1}{x-y}\right)\right] = \boxed{0}$.
  - <u>Example</u>: Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{x^5+y^4}{x^4+y^2}$.
    - We split the limit apart, and compute $\lim_{(x,y)\to(0,0)} \dfrac{x^5}{x^4+y^2}$ and $\lim_{(x,y)\to(0,0)} \dfrac{y^4}{x^4+y^2}$ separately.
    - We have $\left|\dfrac{x^5}{x^4+y^2}\right| \le \left|\dfrac{x^5}{x^4}\right| = |x|$, so since $\lim_{(x,y)\to(0,0)} |x| \to 0$, by the squeeze rule we see $\lim_{(x,y)\to(0,0)} \dfrac{x^5}{x^4+y^2} = 0$.
    - Similarly, $\left|\dfrac{y^4}{x^4+y^2}\right| \le \left|\dfrac{y^4}{y^2}\right| = \left|y^2\right|$, so since $\lim_{(x,y)\to(0,0)} \left|y^2\right| \to 0$, by the squeeze rule we see $\lim_{(x,y)\to(0,0)} \dfrac{y^4}{x^4+y^2} = 0$.
    - Thus, by the sum rule, we see $\lim_{(x,y)\to(0,0)} \dfrac{x^5+y^4}{x^4+y^2} = \boxed{0}$.

- Using the limit rules and some of the basic limit evaluations, we can establish that the usual slate of functions is continuous:
  - Any polynomial in x and y is continuous everywhere.

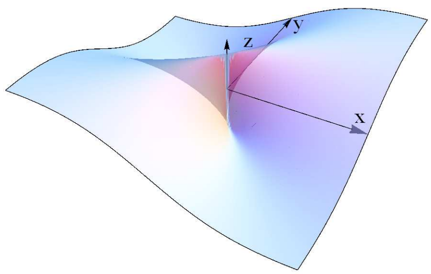
- Any quotient of polynomials $\dfrac{p(x,y)}{q(x,y)}$ is continuous everywhere that the denominator is nonzero.
- The exponential, sine, and cosine of any continuous function are all continuous everywhere.
- The logarithm of a positive continuous function is continuous.

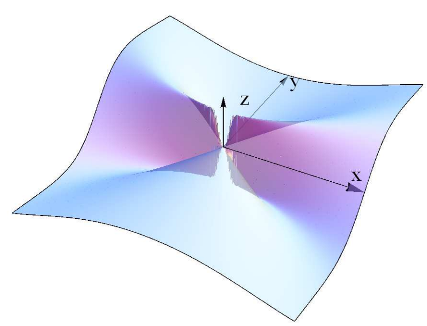
- For one-variable limits, we also have a notion of "one-sided" limits, namely, the limits that approach the target point either from above or from below. In the multiple-variable case, there are many more paths along which we can approach our target point.
  - For example, if our target point is the origin $(0,0)$, then we could approach along the positive x-axis, or the positive y-axis, or along any line through the origin... or along the curve $y = x^2$, or any other continuous curve that approaches the origin.

- As with limits in a single variable, if limits from different directions have different values, then the overall limit does not exist. More precisely:

- <u>Test</u> ("Two Paths Test"): Let $f(x,y)$ be a function of two variables and (a,b) be a point. If there are two continuous paths passing through the point (a,b) such that f has different limits as $(x,y) \to (a,b)$ along the two paths, then $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.
  - The proof is essentially just the definition of limit: if the limit exists, then f must stay within a very small range near (a,b). But if there are two paths through (a,b) along which f approaches different values, then the values of f near (a,b) do not stay within a small range.
  - To apply the test, we try to generate some parametric curves $(x(t), y(t))$ that pass through the point (a,b) when $t = t_0$, and compute $\displaystyle\lim_{t\to t_0} f\left(x(t),\, y(t)\right)$ as a limit in the single variable t.

- <u>Example</u>: Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.
  - We try some simple paths: along the path $(x,y) = (t,0)$ as $t \to 0$ the limit becomes $\displaystyle\lim_{t\to 0} \frac{t\cdot 0}{t^2+0^2} = \lim_{t\to 0} 0 = 0$.
  - Along the path $(x,y) = (0,t)$ as $t \to 0$ we have $\displaystyle\lim_{t\to 0} \frac{0\cdot t}{0^2+t^2} = \lim_{t\to 0} 0 = 0$.
  - Along these two paths the limits are equal. But this does not show the existence of the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.
  - Let's try along the path $(x,y) = (t,t)$: the limit then becomes $\displaystyle\lim_{t\to 0} \frac{t^2}{t^2+t^2} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}$.
  - Thus, along the two paths $(x,y) = (0,t)$ and $(x,y) = (t,t)$, the function has different limits as $(x,y) \to (0,0)$. Hence the limit does not exist.
  - Below on the left is a plot of the function near $(0,0)$:





- <u>Example</u>: Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ does not exist. (Plotted above, on the right.)
  - Along $(x,y) = (0,t)$ the limit is $\displaystyle\lim_{t\to 0} \frac{0\cdot t}{0+t^2} = \lim_{t\to 0} \frac{0}{t^2} = 0$.

- Along $(x, y) = (t, t^2)$ the limit is $\lim_{t\to 0} \frac{t^2 \cdot t^2}{t^4 + t^4} = \lim_{t\to 0} \frac{t^4}{2t^4} = \frac{1}{2}$.
- Along these two paths the limit has different values, so the general limit does not exist.
- Remark: If we try $(x, y) = (t, mt)$ for an arbitrary m, we get $\lim_{t\to 0} \frac{t(mt)^2}{t^2 + (mt)^4} = \lim_{t\to 0} \frac{m^2t^3}{t^2 + m^4t^4} = \lim_{t\to 0} \frac{m^2t}{1 + m^4t^2} = 0$, since $\frac{m^2t}{1 + m^4t^2} \to 0$ as $t \to 0$. So we see that along every line through the origin (including $x = 0$, which we analyzed above), the function has the limit 0. Nevertheless, the limit does not exist!

- Our discussion so far has dealt with functions of two variables. Everything we have done can be generalized in a natural way to functions of more than two variables.
  - However, since we will not really use limits in a major way, we will omit a detailed discussion of limits involving functions of more than two variables.

2.1.2 Partial Derivatives

- Partial derivatives are simply the usual notion of differentiation applied to functions of more than one variable. However, since we now have more than one variable, we also have more than one natural way to compute a derivative.

- Definition: For a function $f(x, y)$ of two variables, we define the partial derivative of f with respect to x as $\frac{\partial f}{\partial x} = f_x = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}$ and the partial derivative of f with respect to y as $\frac{\partial f}{\partial y} = f_y = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}$.
  - Notation: In multivariable calculus, we use the symbol ∂ (typically pronounced either like the letter d or as "del") to denote taking a derivative, in contrast to single-variable calculus where we use the symbol d.
  - We will frequently use both notations $\frac{\partial f}{\partial y}$ and f_y to denote partial derivatives: we generally use the difference quotient notation when we want to emphasize a formal property of a derivative, and the subscript notation when we want to save space.
  - Geometrically, the partial derivative f_x captures how fast the function f is changing in the x-direction, and f_y captures how fast f is changing in the y-direction.

- To evaluate a partial derivative of the function f with respect to x, we need only pretend that all the other variables (i.e., everything except x) that f depends on are constants, and then just evaluate the derivative of f with respect to x as a normal one-variable derivative.
  - All of the derivative rules (the product rule, quotient rule, chain rule, etc.) from one-variable calculus still hold: there will just be extra variables floating around.

- Example: Find f_x and f_y for $f(x, y) = x^3y^2 + e^x$.
  - For f_x, we treat y as a constant and x as the variable. Thus, we see that $f_x = \boxed{3x^2 \cdot y^2 + e^x}$.
  - Similarly, to find f_y, we instead treat x as a constant and y as the variable, to get $f_y = \boxed{x^3 \cdot 2y + 0 = 2x^3y}$. (Note in particular that the derivative of e^x with respect to y is zero.)

- Example: Find f_x and f_y for $f(x, y) = \ln(x^2 + y^2)$.
  - For f_x, we treat y as a constant and x as the variable. We can apply the chain rule to get $f_x = \boxed{\frac{2x}{x^2 + y^2}}$, since the derivative of the inner function $x^2 + y^2$ with respect to x is $2x$.
  - Similarly, we can use the chain rule to find the partial derivative $f_y = \boxed{\frac{2y}{x^2 + y^2}}$.

- Example: Find f_x and f_y for $f(x,y) = \dfrac{e^{x\,y}}{x^2+x}$.
  - For f_x we apply the quotient rule: $f_x = \dfrac{\frac{\partial}{\partial x}\left[e^{xy}\right] \cdot (x^2+x) - e^{xy} \cdot \frac{\partial}{\partial x}\left[x^2+x\right]}{(x^2+x)^2}$. Then we can evaluate the derivatives in the numerator to get $f_x = \boxed{\dfrac{(y\,e^{xy}) \cdot (x^2+x) - e^{xy} \cdot (2x+1)}{(x^2+x)^2}}$.
  - For f_y, the calculation is easier because the denominator is not a function of y. So in this case, we just need to use the chain rule to see that $f_y = \boxed{\dfrac{1}{x^2+x} \cdot (x\,e^{xy})}$.
- We can generalize partial derivatives to functions of more than two variables: for each input variable, we get a partial derivative with respect to that variable. The procedure remains the same: treat all variables except the variable of interest as constants, and then differentiate with respect to the variable of interest.
- Example: Find f_x, f_y, and f_z for $f(x,y,z) = y\,z\,e^{2x^2-y}$.
  - By the chain rule we have $f_x = \boxed{y\,z \cdot e^{2x^2-y} \cdot 4x}$. (We don't need the product rule for f_x since y and z are constants.)
  - For f_y we need to use the product rule since f is a product of two nonconstant functions of y. We get $f_y = z \cdot e^{2x^2-y} + y\,z \cdot \frac{\partial}{\partial y}\left[e^{2x^2-y}\right]$, and then using the chain rule gives $f_y = \boxed{z\,e^{2x^2-y} - y\,z \cdot e^{2x^2-y}}$.
  - For f_z, all of the terms except for z are constants, so we have $f_z = \boxed{y\,e^{2x^2-y}}$.
- Like in the one-variable case, we also have higher-order partial derivatives, obtained by taking a partial derivative of a partial derivative.
  - For a function of two variables, there are four second-order partial derivatives $f_{xx} = \frac{\partial}{\partial x}[f_x]$, $f_{xy} = \frac{\partial}{\partial y}[f_x]$, $f_{yx} = \frac{\partial}{\partial x}[f_y]$, and $f_{yy} = \frac{\partial}{\partial y}[f_y]$.
  - Remark: Partial derivatives in subscript notation are applied left-to-right, while partial derivatives in differential operator notation are applied right-to-left. (In practice, the order of the partial derivatives rarely matters, as we will see.)
- Example: Find the second-order partial derivatives f_{xx}, f_{xy}, f_{yx}, and f_{yy} for $f(x,y) = x^3y^4 + y\,e^{2x}$.
  - First, we have $f_x = 3x^2y^4 + 2y\,e^{2x}$ and $f_y = 4x^3y^3 + e^{2x}$.
  - Then we have $f_{xx} = \frac{\partial}{\partial x}\left[3x^2y^4 + 2y\,e^{2x}\right] = \boxed{6xy^4 + 4y\,e^{2x}}$ and $f_{xy} = \frac{\partial}{\partial y}\left[3x^2y^4 + 2y\,e^{2x}\right] = \boxed{12x^2y^3 + 2e^{2x}}$.
  - Also we have $f_{yx} = \frac{\partial}{\partial x}\left[4x^3y^3 + e^{2x}\right] = \boxed{12x^2y^3 + 2e^{2x}}$ and $f_{yy} = \frac{\partial}{\partial y}\left[4x^3y^3 + e^{2x}\right] = \boxed{12x^3y^2}$.
- Notice that $f_{xy} = f_{yx}$ for the function in the example above. This is not an accident:
- Theorem (Clairaut): If both partial derivatives f_{xy} and f_{yx} are continuous, then they are equal.
  - In other words, these "mixed partials" are always equal (given mild assumptions about continuity), so there are really only three second-order partial derivatives.
  - This theorem can be proven using the limit definition of derivative and the Mean Value Theorem, but the details are unenlightening, so we will omit them.
- We can continue on and take higher-order partial derivatives. For example, a function $f(x,y)$ has eight third-order partial derivatives: f_{xxx}, f_{xxy}, f_{xyx}, f_{xyy}, f_{yxx}, f_{yxy}, f_{yyx}, and f_{yyy}.
  - By Clairaut's Theorem, we can reorder the partial derivatives any way we want (if they are continuous, which is almost always the case). Thus, $f_{xxy} = f_{xyx} = f_{yxx}$, and $f_{xyy} = f_{yxy} = f_{yyx}$.
  - So in fact, $f(x,y)$ only has four different third-order partial derivatives: f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy}.

- Example: Find the third-order partial derivatives f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy} for $f(x, y) = x^4y^2 + x^3e^y$.
  - First, we have $f_x = 4x^3y^2 + 3x^2e^y$ and $f_y = 2x^4y + x^3e^y$.
  - Next, $f_{xx} = 12x^2y^2 + 6xe^y$, $f_{xy} = 8x^3y + 3x^2e^y$, and $f_{yy} = 2x^4 + x^3e^y$.
  - Finally, $f_{xxx} = \boxed{24xy^2 + 6e^y}$, $f_{xxy} = \boxed{24x^2y + 6xe^y}$, $f_{xyy} = \boxed{8x^3 + 3x^2e^y}$, and $f_{yyy} = \boxed{x^3e^y}$.
- Example: If all 5th-order partial derivatives of $f(x, y, z)$ are continuous and $f_{xyz} = e^{xyz}$, what is f_{zzyyx}?
  - By Clairaut's theorem, we can differentiate in any order, and so $f_{zzyyx} = f_{xyzyz} = (f_{xyz})_{yz}$.
  - Since $f_{xyz} = e^{xyz}$ we obtain $(f_{xyz})_y = xze^{xyz}$ and then $(f_{xyz})_{yz} = \boxed{xe^{xyz} + x^2yze^{xyz}}$.

2.2 Directional Derivatives, Gradients, and Tangent Planes

- In this section, we generalize our definition of the partial derivative to allow us to calculate derivatives in any direction, not just the coordinate directions.
  - Although this is a more general idea than the notion of partial derivative, it turns out in fact that we can compute these general directional derivatives using the partial derivatives and an associated vector known as the gradient.
  - We can also give a convenient procedure for finding the tangent line to an implicit curve, and the tangent plane to an implicit surface, using the gradient, so we will discuss this topic as well.

2.2.1 Directional Derivatives

- Aside from the partial derivatives, there is another notion of derivative for a function of several variables. If, for example, $f(x, y)$ is a function of two variables, then the partial derivatives f_x and f_y measure how much f is changing in the x-direction and in the y-direction, respectively.
  - However, there is no reason we couldn't ask for the rate of change of f in *any* direction, not just one of the coordinate directions.
  - We can, in fact, pose this precise question using what is called the directional derivative:
- Definition: If $\mathbf{v} = \langle v_x, v_y \rangle$ is a unit vector, then the directional derivative of $f(x, y)$ in the direction of $\mathbf{v}$ at (x, y), denoted $D_{\mathbf{v}}(f)(x, y)$, is defined to be the limit $D_{\mathbf{v}}(f)(x, y) = \lim_{h \to 0} \frac{f(x + h\,v_x,\, y + h\,v_y) - f(x, y)}{h}$, provided that the limit exists.
  - The corresponding definition when f has more variables is analogous.
  - Important Note: In the definition of directional derivative, the vector $\mathbf{v}$ is a *unit vector*. We sometimes will speak of the directional derivative of a function in the direction of a vector $\mathbf{w}$ whose length is not 1: what we mean by this is the directional derivative in the direction of the unit vector $\frac{\mathbf{w}}{||\mathbf{w}||}$ pointing the same direction as $\mathbf{w}$.
  - The limit in the definition is explicit, but a little bit hard to understand. If we write things in vector notation, with $\mathbf{x} = \langle x, y \rangle$, then the definition is clearer: it becomes $D_{\mathbf{v}}(f)(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\,\mathbf{v}) - f(\mathbf{x})}{h}$. (Compare it to the definition of the derivative of a function of one variable.)
- When $\mathbf{v}$ is the unit vector in one of the coordinate directions, the directional derivative reduces to the corresponding partial derivative. Explicitly:
  - If $\mathbf{v} = \langle 1, 0 \rangle$, the unit vector in the x-direction, then $D_{\mathbf{v}}(f)(x, y) = \lim_{h \to 0} \frac{f(x + h,\, y) - f(x, y)}{h} = f_x(x, y)$.
  - If $\mathbf{v} = \langle 0, 1 \rangle$, the unit vector in the y-direction, then $D_{\mathbf{v}}(f)(x, y) = \lim_{h \to 0} \frac{f(x,\, y + h) - f(x, y)}{h} = f_y(x, y)$.

- It is cumbersome to apply the definition to compute directional derivatives. Fortunately, there is an easier way to compute them, using partial derivatives and a vector called the gradient.

- Definition: The gradient of a function $f(x, y)$, denoted ∇f, is the vector-valued function $\nabla f(x, y) = \langle f_x(x, y),\ f_y(x, y)\rangle$. For a function $g(x, y, z)$, the gradient ∇g is $\nabla g(x, y, z) = \langle g_x(x, y, z),\ g_y(x, y, z),\ g_z(x, y, z)\rangle$.
  - Note: The symbol ∇ is called "nabla", and is pronounced either as "nabla" or as "del".
  - If f is a function of some other number of variables, the gradient is defined analogously: the component of the vector in any coordinate $\star$ is the partial derivative $f_\star$ of f with respect to $\star$.
  - Note that the gradient of f is a vector-valued function: it takes the same number of arguments as f does, and outputs a vector in the same number of coordinates.
  - Example: For $f(x, y, z) = x^2 + y^2 + z^2$, we have $\nabla f(x, y, z) = \langle f_x, f_y, f_z\rangle = \boxed{\langle 2x, 2y, 2z\rangle}$.
  - Example: For $f(x, y) = x^2 \cos(y)$, we have $\nabla f(x, y) = \langle f_x, f_y\rangle = \boxed{\left\langle 2x\,\cos(y),\ -x^2\sin(y)\right\rangle}$.

- Theorem (Gradient and Directional Derivatives): If $\mathbf{v}$ is any unit vector, and f is a function all of whose partial derivatives are continuous, then the directional derivative $D_{\mathbf{v}} f$ satisfies $D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v}$. In other words, the directional derivative of f in the direction of $\mathbf{v}$ is equal to the dot product of ∇f with the vector $\mathbf{v}$.
  - Warning: This result requires the direction $\mathbf{v}$ to be a unit vector. If the desired direction is not a unit vector, it is necessary to normalize the direction vector first!
  - The proof for functions of more than two variables is essentially the same as for functions of two variables. Thus, for ease of notation, we will only give the proof for functions of two variables.
  - Proof (two-variable case): If $v_x = 0$ then the directional derivative is the x-partial (or its negative), and if $v_y = 0$ then the directional derivative is the y-partial (or its negative), and the result is true.
  - If v_x and v_y are both nonzero, then we can write

$$\begin{aligned}
D_{\mathbf{v}}(f)(x, y) &= \lim_{h\to 0} \frac{f(x + h\,v_x,\ y + h\,v_y) - f(x, y)}{h}, \\
&= \lim_{h\to 0} \left[\frac{f(x + h\,v_x,\ y + h\,v_y) - f(x, y + h\,v_y)}{h} + \frac{f(x, y + h\,v_y) - f(x, y)}{h}\right] \\
&= \lim_{h\to 0} \left[\frac{f(x + h\,v_x,\ y + h\,v_y) - f(x, y + h\,v_y)}{h\,v_x}\,v_x + \frac{f(x, y + h\,v_y) - f(x, y)}{h\,v_y}\,v_y\right] \\
&= v_x \lim_{h\to 0} \left[\frac{f(x + h\,v_x,\ y + h\,v_y) - f(x, y + h\,v_y)}{h\,v_x}\right] + v_y \lim_{h\to 0} \left[\frac{f(x, y + h\,v_y) - f(x, y)}{h\,v_y}\right]
\end{aligned}$$

  - By continuity, one can check that the quotient $\left[\dfrac{f(x + h\,v_x,\ y + h\,v_y) - f(x, y + h\,v_y)}{h\,v_x}\right]$ tends to the partial derivative $\dfrac{\partial f}{\partial x}(x, y)$ as $h \to 0$, and that the second quotient $\left[\dfrac{f(x, y + h\,v_y) - f(x, y)}{h\,v_y}\right]$ equals the partial derivative $\dfrac{\partial f}{\partial y}(x, y)$ as $h \to 0$.
  - Plugging these values in yields $D_{\mathbf{v}}(f)(x, y) = v_x \dfrac{\partial f}{\partial x} + v_y \dfrac{\partial f}{\partial y} = \langle v_x, v_y\rangle \cdot \langle f_x, f_y\rangle = \mathbf{v} \cdot \nabla f$, as claimed.

- Example: If $\mathbf{v} = \left\langle \dfrac{3}{5}, \dfrac{4}{5}\right\rangle$, and $f(x, y) = 2x + y$, find the directional derivative of f in the direction of $\mathbf{v}$ at $(1, 2)$, both from the definition of directional derivative and from the gradient theorem.
  - Since $\mathbf{v}$ is a unit vector, the definition says that

$$\begin{aligned}
D_{\mathbf{v}}(f)(1, 2) &= \lim_{h\to 0} \frac{f(x + h\,v_x,\ y + h\,v_y) - f(x, y)}{h} = \lim_{h\to 0} \frac{f(1 + \frac{3}{5}h,\ 2 + \frac{4}{5}h) - f(x, y)}{h} \\
&= \lim_{h\to 0} \frac{\left[2(1 + \frac{3}{5}h) + (2 + \frac{4}{5}h)\right] - [2 \cdot 1 + 2]}{h} = \lim_{h\to 0} \frac{(4 + 2h) - 4}{h} = \boxed{2}.
\end{aligned}$$

- To compute the answer using the gradient, we immediately have $f_x = 2$ and $f_y = 1$, so $\nabla f = \langle 2, 1 \rangle$. Then the theorem says $D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} = \langle 2, 1 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \frac{6}{5} + \frac{4}{5} = \boxed{2}$.
- Observe how much easier it was to use the gradient to compute the directional derivative!

- Example: Find the rate of change of the function $f(x, y, z) = e^{xyz}$ at the point $(x, y, z) = (1, 1, 1)$ in the direction of the vector $\mathbf{w} = \langle -2, 1, 2 \rangle$.
  - Note that $\mathbf{w}$ is not a unit vector, so we must normalize it: since $||\mathbf{w}|| = \sqrt{(-2)^2 + 1^2 + 2^2} = 3$, we take $\mathbf{v} = \frac{\mathbf{w}}{||\mathbf{w}||} = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$.
  - Furthermore, we compute $\nabla f = \langle yz\, e^{xyz},\ xz\, e^{xyz},\ xy\, e^{xyz} \rangle$, so $\nabla f(-2, 1, 2) = \langle 2e^{-4}, -4e^{-4}, -2e^{-4} \rangle$.
  - Then the desired rate of change is $D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} = -\frac{2}{3}(2e^{-4}) + \frac{1}{3}(-4e^{-4}) + \frac{2}{3}(-2e^{-4}) = \boxed{-4e^{-4}}$.

- From the gradient theorem for computing directional derivatives, we can deduce several corollaries about how the magnitude of the directional derivative depends on the direction $\mathbf{v}$:

- Corollaries: Suppose f is a differentiable function with gradient ∇f and $\mathbf{v}$ is a unit vector. Then the following hold:
  1. The maximum value of $D_{\mathbf{v}} f$ occurs when $\mathbf{v}$ is a unit vector pointing in the direction of ∇f, if $\nabla f \neq \mathbf{0}$, and the maximum value is $||\nabla f||$. In other words, the gradient points in the direction where f is increasing most rapidly.
  2. The minimum value of $D_{\mathbf{v}} f$ occurs when $\mathbf{v}$ is a unit vector pointing the opposite direction of ∇f, if $\nabla f \neq \mathbf{0}$, and the minimum value is $-||\nabla f||$. In other words, the gradient points in the opposite direction from where f is decreasing most rapidly.
  3. The value of $D_{\mathbf{v}} f$ is zero if and only if $\mathbf{v}$ is orthogonal to the gradient ∇f.
     - Proof: If $\mathbf{v}$ is a unit vector, then the directional derivative satisfies $D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} = ||\nabla f||\ ||\mathbf{v}|| \cos(\theta)$, where θ is the angle between ∇f and $\mathbf{v}$.
     - We know that $||\nabla f||$ is a fixed nonnegative number, and $||\mathbf{v}|| = 1$. So if we change the direction of $\mathbf{v}$, the only quantity in $||\nabla f||\ ||\mathbf{v}|| \cos(\theta)$ that changes is $\cos(\theta)$.
     - So, for (1), the maximum value of $\nabla f \cdot \mathbf{v}$ occurs when $\cos(\theta) = 1$, which is to say, when ∇f and $\mathbf{v}$ are parallel and point in the same direction. The maximum value is then just $||\nabla f||$.
     - For (2), the minimum value of $\nabla f \cdot \mathbf{v}$ occurs when $\cos(\theta) = -1$, which is to say, when ∇f and $\mathbf{v}$ are parallel and point in opposite directions. The minimum value is then just $-||\nabla f||$.
     - Finally, for (3), $D_v f$ is zero if and only if $\nabla f \cdot \mathbf{v} = 0$, which is equivalent to saying that ∇f and $\mathbf{v}$ are orthogonal.

- Example: For the function $f(x, y) = x^2 + y^2$, in which direction is f increasing the fastest at $(x, y) = (3, 4)$, and how fast is it increasing? In which direction is f decreasing the fastest, and how fast?
  - The corollaries tell us that the function is increasing the fastest in the direction of the gradient and decreasing the fastest in the direction opposite to the gradient, and the corresponding maximum rate of increase (or decrease) is the magnitude of the gradient.
  - We have $f_x = 2x$ and $f_y = 2y$ so $\nabla f = \langle 2x, 2y \rangle$, and $\nabla f(3, 4) = \langle 6, 8 \rangle$.
  - Since $||\nabla f(3, 4)|| = \sqrt{6^2 + 8^2} = 10$, we see that the maximum value of the directional derivative $D_{\mathbf{v}} f$ is $\boxed{10}$ and occurs in the direction of $\boxed{\frac{\langle 6, 8 \rangle}{10} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle}$.
  - Likewise, the minimum value of $D_{\mathbf{v}} f$ is $\boxed{-10}$ and occurs in the direction of $\boxed{-\frac{\langle 6, 8 \rangle}{10} = \left\langle -\frac{3}{5}, -\frac{4}{5} \right\rangle}$.

- Example: For $f(x, y, z) = x^3 + y^3 + 2z^3$, in which direction is f increasing the fastest at $(x, y, z) = (2, -2, 1)$, and how fast is it increasing? In which direction is f decreasing the fastest, and how fast?

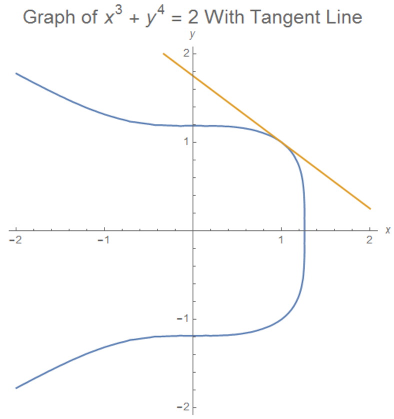
- As above, f is increasing the fastest in the direction of ∇f and decreasing the fastest in the direction opposite ∇f , and the corresponding maximum rate of increase (or decrease) is $||\nabla f||$.
- We have $\nabla f(x,y,z) = \langle 3x^2,\, 3y^2,\, 6z^2 \rangle$, so $\nabla f(2,-2,1) = \langle 12,12,6 \rangle$.
- Since $||\nabla f(2,-2,1)|| = \sqrt{12^2+12^2+6^2} = 18$, we see that the maximum value of the directional derivative $D_{\mathbf{v}} f$ is $\boxed{18}$ and occurs in the direction of $\boxed{\frac{\langle 12,12,6 \rangle}{18} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle}$.
- Likewise, the minimum value of $D_{\mathbf{v}} f$ is $\boxed{-18}$ and occurs in the direction of $\boxed{-\frac{\langle 12,12,6 \rangle}{18} = \left\langle -\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3} \right\rangle}$.

- Example: For $f(x,y) = \ln(x^2+xy+1)$, find a unit vector direction in which the value of f is not changing at the point $(1,1)$.
  - From our results above, we know that the directions in which f are not changing are those orthogonal to the gradient ∇f.
  - Here, we have $\nabla f(x,y) = \left\langle \frac{2x+y}{x^2+xy+1}, \frac{x}{x^2+xy+1} \right\rangle$, so $\nabla f(1,1) = \left\langle 1, \frac{1}{3} \right\rangle$.
  - Thus, the vector $\langle a,b \rangle$ will be orthogonal to ∇f precisely when $\langle a,b \rangle \cdot \left\langle 1, \frac{1}{3} \right\rangle = 0$, which is to say, when $a + b/3 = 0$, so that $b = -3a$.
  - The desired vectors are thus of the form $\langle a, -3a \rangle = a \langle 1, -3 \rangle$. Normalizing shows that there are two such unit vectors: $\boxed{\frac{1}{\sqrt{10}} \langle 1,-3 \rangle}$ and its negative $\boxed{\frac{1}{\sqrt{10}} \langle -1,3 \rangle}$.

2.2.2 Tangent Lines and Tangent Planes

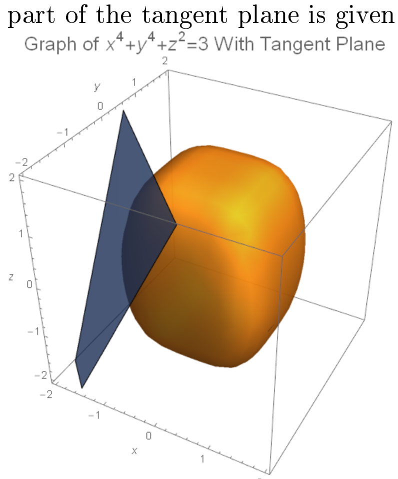
- For a function $f(x)$ of a single variable, one of the fundamental uses of the derivative is to find an equation for the line tangent to the graph of $y = f(x)$ at a point $(a, f(a))$. For functions of more than one variable, we have a closely-related notion of a tangent line to a curve and of a tangent plane to a surface.
  - Per our discussion of the gradient (part 3 of the corollary above), if we plot a level curve $f(x,y) = d$, then at any point (a,b) on the curve, the gradient $\nabla f(a,b)$ at that point will be a normal vector to the graph of the level curve at (a,b).
  - This is true because, by the definition of the level curve $f(x,y) = d$, the tangent vector to the level curve at (a,b) is a direction in which f is not changing.
  - Thus, we can see that the tangent line to the curve $f(x,y) = d$ at (a,b) has equation $\nabla f(a,b) \cdot \langle x-a, y-b \rangle = 0$, or, explicitly, $f_x(a,b) \cdot (x-a) + f_y(a,b) \cdot (y-b) = 0$.
  - Note that we could also have calculated the slope of this tangent line via implicit differentiation. Both methods, of course, will give the same answer.

- Example: Find an equation for the tangent line to the curve $x^3 + y^4 = 2$ at the point $(1,1)$.
  - This curve is the level set $f(x,y) = 2$ for $f(x,y) = x^3 + y^4$, and we have $(a,b) = (1,1)$.
  - We have $f_x = 3x^2$ and $f_y = 4y^3$ so $\nabla f = \langle 3x^2, 4y^3 \rangle$, and $\nabla f(1,1) = \langle 3,4 \rangle$.
  - Therefore, an equation for the tangent line is $\boxed{3(x-1) + 4(y-1) = 0}$.
  - A plot of the curve together with this tangent line is given below:



- Similarly, if we plot a level surface $f(x, y, z) = d$, then at any point on the surface, the gradient ∇f gives a vector normal to that surface.
  - Recall that the equation of the plane with normal vector $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ passing through (a, b, c) is $v_x(x - a) + v_y(y - b) + v_z(z - c) = 0$.
  - Thus, the tangent plane to the surface $f(x, y, z) = d$ at the point (a, b, c) has equation $\nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0$.
  - Explicitly, the tangent plane is $f_x(a, b, c) \cdot (x - a) + f_y(a, b, c) \cdot (y - b) + f_z(a, b, c) \cdot (z - c) = 0$.
  - The line in the direction of the normal vector to this plane is called the <u>normal line</u> to the surface. It can be parametrized as $l : \langle x, y, z \rangle = \langle a, b, c \rangle + t\,\nabla f(a, b, c)$.

- <u>Example</u>: Find an equation for the tangent plane to the surface $x^4 + y^4 + z^2 = 3$ at the point $(-1, -1, 1)$.
  - This surface is the level set $f(x, y, z) = 3$ for $f(x, y, z) = x^4 + y^4 + z^2$, and we have $(a, b, c) = (-1, -1, 1)$.
  - We have $f_x = 4x^3$, $f_y = 4y^3$, and $f_z = 2z$ so $\nabla f = \langle 4x^3, 4y^3, 2z \rangle$, and $\nabla f(-1, -1, 1) = \langle -4, -4, 2 \rangle$.
  - Therefore, an equation for the tangent plane is $\boxed{-4(x + 1) - 4(y + 1) + 2(z - 1) = 0}$.
  - A plot of the surface together with part of the tangent plane is given below:



- <u>Example</u>: Find an equation for the tangent plane to $z = \ln(2x^2 - y^2)$ at the point with $(x, y) = (-1, 1)$.
  - Note that we must first rearrange the given equation to have the form $f(x, y, z) = d$.
  - This curve is the level set $f(x, y, z) = 0$ for $f(x, y, z) = \ln(2x^2 - y^2) - z$. When $(x, y) = (-1, 1)$ we see $z = \ln(2 - 1) = 0$, and so $(a, b, c) = (-1, 1, 0)$.
  - We have $f_x = \dfrac{4x}{2x^2 - y^2}$, $f_y = \dfrac{-2y}{2x^2 - y^2}$, and $f_z = -1$ so $\nabla f = \left\langle \dfrac{4x}{2x^2 - y^2}, \dfrac{-2y}{2x^2 - y^2}, -1 \right\rangle$ and $\nabla f(-1, 1, 0) = \langle -4, -2, -1 \rangle$.
  - Therefore, an equation for the tangent plane is $\boxed{-4(x + 1) - 2(y - 1) - (z + 1) = 0}$.

2.3 The Chain Rule

- In many situations, we have functions that depend on variables indirectly, and we often need to determine the precise nature of the dependence of one variable on another. To do this, we want to generalize the chain rule for functions of several variables.

2.3.1 The Multivariable Chain Rule

- With functions of several variables, each of which is defined in terms of other variables (for example, $f(x, y)$ where x and y are themselves functions of s and t), we recover versions of the chain rule specific to the relations between the variables involved.
  - Recall that the chain rule for functions of one variable states that $\frac{dg}{dx} = \frac{dg}{dy} \cdot \frac{dy}{dx}$, if g is a function of y and y is a function of x.
  - The various chain rules for functions of more than one variable have a similar form, but they will involve more terms that depend on the relationships between the variables.

- Here is the rough idea of the formal proof of the chain rule for the situation of finding $\frac{df}{dt}$ for a function $f(x, y)$ where x and y are both functions of t:
  - If t changes to $t + \Delta t$, then x changes to $x + \Delta x$ and y changes to $y + \Delta y$.
  - Then $\Delta f = f(x + \Delta x, y + \Delta y) - f(x, y)$ is roughly equal to the directional derivative of $f(x, y)$ in the direction of the (non-unit) vector $\langle \Delta x, \Delta y \rangle$.
  - From the results about the gradient and directional derivatives, we know that this directional derivative is equal to the dot product of $\langle \Delta x, \Delta y \rangle$ with the gradient $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$.
  - Then $\frac{\Delta f}{\Delta t} \approx \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t} \right\rangle$. Taking the limit as $\Delta t \to 0$ (and verifying a few other details) then gives us that $\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$.

- The simplest method for generating the statement of the chain rule specific to any particular set of dependencies of variables is to draw a "tree diagram" as follows:
  - Step 1: Start with the initial function f, and draw an arrow pointing from f to each of the variables it depends on.
  - Step 2: For each variable listed, draw new arrows branching from that variable to any other variables they depend on. Repeat the process until all dependencies are shown in the diagram.
  - Step 3: Associate each arrow from one variable to another with the derivative $\frac{\partial[\text{top}]}{\partial[\text{bottom}]}$.
  - Step 4: To write the version of the chain rule that gives the derivative $\frac{\partial v_1}{\partial v_2}$ for any variables v_1 and v_2 in the diagram (where v_2 depends on v_1), first find all paths from v_1 to v_2.
  - Step 5: For each path from v_1 to v_2, multiply all of the derivatives that appear in each path from v_1 to v_2. Then sum the results over all of the paths: this is $\frac{\partial v_1}{\partial v_2}$.

- Example: State the chain rule that computes $\frac{df}{dt}$ for the function $f(x, y, z)$, where each of x, y, and z is a function of the variable t.
  - First, we draw the tree diagram: $\begin{array}{ccc} & f & \\ \swarrow & \downarrow & \searrow \\ x & y & z \\ \downarrow & \downarrow & \downarrow \\ t & t & t \end{array}$.

- In the tree diagram, there are 3 paths from f to t: they are $f \to x \to t$, $f \to y \to t$, and $f \to z \to t$.
- The path $f \to x \to t$ gives the product $\dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial t}$, while the path $f \to y \to t$ gives the product $\dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial t}$, and the path $f \to z \to t$ gives the product $\dfrac{\partial f}{\partial z} \cdot \dfrac{\partial z}{\partial t}$.
- The statement of the chain rule here is $\boxed{\dfrac{df}{dt} = \dfrac{\partial f}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dt} + \dfrac{\partial f}{\partial z} \cdot \dfrac{dz}{dt}}$.
- Remark: We write df/dt rather than $\partial f/\partial t$ because f ultimately depends on only the single variable t, so we are actually computing a single-variable derivative and not a partial derivative.

- Example: State the chain rule that computes $\dfrac{\partial f}{\partial t}$ and $\dfrac{\partial f}{\partial s}$ for the function $f(x, y)$, where $x = x(s, t)$ and $y = y(s, t)$ are both functions of s and t.
  - First, we draw the tree diagram: $\begin{array}{ccccccc} & & & f & & & \\ & & \swarrow & & \searrow & & \\ & x & & & & y & \\ \swarrow & \downarrow & & & & \downarrow & \searrow \\ s & t & & & & s & t \end{array}$.
  - In this diagram, there are 2 paths from f to s: they are $f \to x \to s$ and $f \to y \to s$, and also two paths from f to t: $f \to x \to t$ and $f \to y \to t$.
  - The path $f \to x \to t$ gives the product $\dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial t}$, while the path $f \to y \to t$ gives the product $\dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial t}$. Similarly, the path $f \to x \to s$ gives the product $\dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial s}$, while $f \to y \to s$ gives the product $\dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial s}$.
  - Thus, the two statements of the chain rule here are $\boxed{\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial s}}$ and $\boxed{\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial t}}$.

- Once we have the appropriate statement of the chain rule, it is easy to work examples with specific functions.

- Example: For $f(x, y) = x^2 + y^2$, with $x = t^2$ and $y = t^4$, find $\dfrac{df}{dt}$, both directly and via the chain rule.
  - In this instance, the chain rule says that $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dt}$.
  - Computing the derivatives shows $\dfrac{df}{dt} = (2x) \cdot (2t) + (2y) \cdot (4t^3)$.
  - Plugging in $x = t^2$ and $y = t^4$ yields $\dfrac{df}{dt} = (2t^2) \cdot (2t) + (2t^4) \cdot (4t^3) = \boxed{4t^3 + 8t^7}$.
  - To do this directly, we would plug in $x = t^2$ and $y = t^4$: this gives $f(x, y) = t^4 + t^8$, so that $\dfrac{df}{dt} = \boxed{4t^3 + 8t^7}$. (Of course, we obtain the same answer either way!)

- Example: For $f(x, y) = x^2 + y^2$, with $x = s^2 + t^2$ and $y = s^3 + t^4$, find $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$ both directly and via the chain rule.
  - By the chain rule we have $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial s} = (2x) \cdot (2s) + (2y) \cdot (3s^2)$. Plugging in $x = s^2 + t^2$ and $y = s^3 + t^4$ yields $\dfrac{\partial f}{\partial s} = (2s^2 + 2t^2) \cdot (2s) + (2s^3 + 2t^4) \cdot (3s^2) = \boxed{4s^3 + 4st^2 + 6s^5 + 6s^2 t^4}$.
  - We also have $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial s} = (2x) \cdot (2t) + (2y) \cdot (4t^3)$. Plugging in $x = s^2 + t^2$ and $y = s^3 + t^4$ yields $\dfrac{\partial f}{\partial s} = (2s^2 + 2t^2) \cdot (2t) + (2s^3 + 2t^4) \cdot (4t^3) = \boxed{4s^2 t + 4t^3 + 8s^3 t^3 + 8t^7}$.

- To do this directly, we plug in $x = s^2+t^2$ and $y = s^3+t^4$: this gives $f(x,y) = (s^2+t^2)^2+(s^3+t^4)^2 = s^4 + 2s^2t^2+t^4+s^6+2s^3t^4+t^8$, so that $\dfrac{\partial f}{\partial t} = \boxed{4s^2t+4t^3+8s^3t^3+8t^7}$ and $\dfrac{\partial f}{\partial s} = \boxed{4s^3+4st^2+6s^5+6s^2t^4}$.

- Amusingly, we can actually derive the other differentiation rules using the (multivariable) chain rule!
  - To obtain the product rule, let $P(f,g) = fg$, where f and g are both functions of x.
  - Then the chain rule says $\dfrac{dP}{dx} = \dfrac{\partial P}{\partial f}\cdot\dfrac{df}{dx} + \dfrac{\partial P}{\partial g}\cdot\dfrac{dg}{dx} = g\cdot\dfrac{df}{dx} + f\cdot\dfrac{dg}{dx}$.
  - If we rewrite this expression using single-variable notation, it reads as the more familiar $(fg)' = f'g+fg'$, which is, indeed, the product rule.
  - Likewise, if we set $Q(f,g) = f/g$ where f and g are both functions of x, then applying the chain rule gives $\dfrac{dQ}{dx} = \dfrac{\partial P}{\partial f}\cdot\dfrac{df}{dx} + \dfrac{\partial P}{\partial g}\cdot\dfrac{dg}{dx} = \dfrac{1}{g}\cdot\dfrac{df}{dx} - \dfrac{f}{g^2}\cdot\dfrac{dg}{dx} = \dfrac{f'g-fg'}{g^2}$; this is the quotient rule.

2.3.2 Implicit Differentiation

- Using the chain rule for several variables, we can give an alternative way to solve problems involving implicit differentiation from calculus of a single variable.

- Theorem (Implicit Differentiation): Given an implicit relation $f(x,y) = c$, the implicit derivative $\dfrac{dy}{dx}$ is $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{f_x}{f_y}$.
  - Proof: Apply the chain rule to the function $f(x,y) = c$, where y is also a function of x.
  - The tree diagram is $\begin{array}{ccc} & & f \\ & \swarrow & \downarrow \\ x & & y \\ & & \downarrow \\ & & x \end{array}$, so there are two paths from f to x: $f \to x$ with product $\dfrac{\partial f}{\partial x}$, and $f \to y \to x$ with product $\dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dx}$.
  - Thus, the chain rule yields $\dfrac{df}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dx}$.
  - However, because $f(x,y) = c$ is a constant function, we have $\dfrac{df}{dx} = 0$: thus, $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dx} = 0$, and so $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$ as claimed.

- Example: Find y', if $y^2x^3 + y\,e^x = 2$.
  - This is the relation $f(x,y) = 2$, where $f(x,y) = y^2x^3 + y\,e^x$.
  - We have $f_x = 3y^2x^2 + y\,e^x$, and $f_y = 2yx^3 + e^x$, so the formula gives $y' = \dfrac{dy}{dx} = \boxed{-\dfrac{3y^2x^2+y\,e^x}{2yx^3+e^x}}$.
  - We can check this by doing the implicit differentiation directly: we have $\dfrac{d}{dx}\left[y^2x^3 + y\,e^x\right] = 0$. Differentiating the left-hand side yields $\left(2yy'x^3 + 3y^2x^2\right) + (y'\,e^x + y\,e^x) = 0$.
  - Rearranging yields $\left(2yx^3 + e^x\right)y' + \left(3y^2x^2 + y\,e^x\right) = 0$, so $y' = \boxed{-\dfrac{3y^2x^2+y\,e^x}{2yx^3+e^x}}$.

- We can also perform implicit differentiation in the event that there are more than 2 variables: for example, for an implicit relation $f(x,y,z) = c$, we can compute implicit derivatives for any of the 3 variables with respect to either of the others, using a chain rule calculation like in the theorem above.

- The idea is simply to view any variables other than our target variables as constants: thus if we have an implicit relation $f(x,y,z) = c$ and want to compute $\dfrac{\partial z}{\partial x}$, we view the other variable (in this case y) as constant, and then the chain rule says that $\dfrac{\partial z}{\partial x} = -\dfrac{\partial f/\partial x}{\partial f/\partial z} = -\dfrac{f_x}{f_z}$.
- If we wanted to compute $\dfrac{\partial z}{\partial y}$ then we would view x as a constant, and the chain rule would then give $\dfrac{\partial z}{\partial y} = -\dfrac{\partial f/\partial y}{\partial f/\partial z} = -\dfrac{f_y}{f_z}$.
- We can compute $\dfrac{\partial y}{\partial x} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$, $\dfrac{\partial y}{\partial z} = -\dfrac{\partial f/\partial z}{\partial f/\partial y}$, $\dfrac{\partial x}{\partial y} = -\dfrac{\partial f/\partial y}{\partial f/\partial x}$, and $\dfrac{\partial x}{\partial z} = -\dfrac{\partial f/\partial z}{\partial f/\partial x}$ in a similar manner.

- <u>Example</u>: Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial y}{\partial z}$ given that $x^2yz + e^x\cos(y) = 3$.
  - The implicit relation is $f(x,y,z) = 3$ where $f(x,y,z) = x^2yz + e^x\cos(y)$.
  - We have $f_x = 2xyz + e^x\cos(y)$, $f_y = x^2z - e^x\sin(y)$, and $f_z = x^2y$.
  - By the implicit differentiation formulas above, we get $\dfrac{\partial z}{\partial x} = -\dfrac{\partial f/\partial x}{\partial f/\partial z} = -\dfrac{f_x}{f_z} = \boxed{-\dfrac{2xyz + e^x\cos(y)}{x^2y}}$, and $\dfrac{\partial y}{\partial z} = -\dfrac{\partial f/\partial z}{\partial f/\partial y} = -\dfrac{f_z}{f_y} = \boxed{-\dfrac{x^2y}{x^2z - e^x\sin(y)}}$.

2.4 Linearization and Taylor Series

- Like with functions of a single variable, we have a notion of a "best linear approximation" and a "best polynomial approximation" to a function of several variables. These ideas are often needed in applications in physics and applied mathematics, when it is necessary to analyze a complicated system with a series of approximations.

2.4.1 Linearization

- If we have a differentiable function $f(x)$ of one variable, we can use derivatives to construct a linear function that is a good approximation to the value of $f(x)$ near a given point.
  - Specifically, we can use the tangent line to the graph of $y = f(x)$ at $x = a$ for this purpose. (Recall that the equation of this tangent line is $y = f(a) + f'(a)\cdot(x-a)$, since it passes through the point $(a, f(a))$ and its slope is $f'(a)$.)
  - Indeed, one can show that the linearization $L(x) = f(a) + f'(a)\cdot(x-a)$ is the only linear function with the property is the only linear function satisfying $\lim\limits_{x\to a}\dfrac{f(x) - L(x)}{x-a} = 0$, which is to say, that the error in approximating $f(x)$ by $L(x)$ drops to zero more rapidly than $|x-a|$ does as $x \to a$.

- If we follow the natural analogy for functions of 2 variables, we should therefore expect that the tangent plane to the graph $z = f(x,y)$ of a differentiable function $f(x,y)$ should serve as the best linear approximation to f near the point of tangency (a,b). Here is a justification of this idea:
  - Suppose we want to compute the change in a function $f(x,y)$ as we move from (a,b) to a nearby point $(a+\Delta x,\, b+\Delta y)$.
  - A slight modification of the definition of the directional derivative says that, for $\mathbf{v} = \langle \Delta x,\, \Delta y\rangle$, we have $||\mathbf{v}||\, D_{\mathbf{v}}f(a,b) = \lim\limits_{h\to 0}\dfrac{f(a+h\Delta x,\, b+h\Delta y) - f(a,b)}{h}$. (If Δx and Δy are small, then the difference quotient should be close to the limit value.)
  - From the properties of the gradient, we know $D_{\mathbf{v}}f(a,b) = \nabla f(a,b)\cdot\dfrac{\mathbf{v}}{||\mathbf{v}||} = \dfrac{1}{||\mathbf{v}||}\left[f_x(a,b)\Delta x + f_y(a,b)\Delta y\right]$.

- Plugging this in eventually gives $f(a+\Delta x, b+\Delta y) \approx f(a,b) + f_x(a,b) \cdot \Delta x + f_y(a,b) \cdot \Delta y$.
- If we write $x = a + \Delta x$ and $y = a + \Delta y$, we see that $f(x,y)$ is approximately equal to the linear function $L(x,y) = f(a,b) + f_x(a,b) \cdot (x-a) + f_y(a,b) \cdot (y-b)$ when $x-a$ and $y-b$ are small.

- Definition: If $f(x,y)$ is a differentiable function of two variables, its linearization at the point (a,b) is the linear function $L(x,y) = f(a,b) + f_x(a,b) \cdot (x-a) + f_y(a,b) \cdot (y-b)$.
  - As predicted above, the linearization is exactly the approximation given by the tangent plane, since the tangent plane to $z = f(x,y)$ at (a,b) has equation $z = f(a,b) + f_x(a,b) \cdot (x-a) + f_y(a,b) \cdot (y-b)$.
  - The idea, as also noted above, is that when $x-a$ and $y-b$ are small, we can use the linearization to estimate the value $f(x,y) \approx L(x,y)$.

- Example: Find the linearization of $f(x,y) = e^{x+y}$ near $(x,y) = (0,0)$, and use it to estimate $f(0.1, 0.1)$.
  - Here, we have $(a,b) = (0,0)$, and we calculate $f_x = e^{x+y}$ and $f_y = e^{x+y}$.
  - Therefore, $L(x,y) = f(0,0) + f_x(0,0) \cdot (x-0) + f_y(0,0) \cdot (y-0) = \boxed{1+x+y}$.
  - The approximate value of $f(0.1, 0.1)$ is then $L(0.1, 0.1) = \boxed{1.2}$. (The actual value is $e^{0.2} \approx 1.2214$, which is reasonably close.)

- We can extend these ideas to linearizations of functions of 3 or more variables:

- Definition: If $f(x,y,z)$ is a differentiable function of three variables, its linearization at the point (a,b,c) is the linear function $L(x,y,z) = f(a,b,c) + f_x(a,b,c) \cdot (x-a) + f_y(a,b,c) \cdot (y-b) + f_z(a,b,c) \cdot (z-c)$.
  - Note here that the formula for the linearization $L(x,y,z)$ at (a,b,c) simply gains the corresponding term for each extra variable.

- Example: Find the linearization of $f(x,y,z) = x^2y^3z^4$ near $(x,y,z) = (1,1,1)$, and use it to estimate $f(1.2, 1.1, 0.9)$.
  - In this case, we have $(a,b,c) = (1,1,1)$, and we calculate $f_x = 2xy^3z^4$, $f_y = 3x^2y^2z^4$, and $f_z = 4x^2y^3z^4$.
  - Then we obtain $L(x,y,z) = f(1,1,1) + f_x(1,1,1) \cdot (x-1) + f_y(1,1,1) \cdot (y-1) + f_z(1,1,1) \cdot (z-1) = \boxed{1 + 2(x-1) + 3(y-1) + 4(z-1)}$.
  - The approximate value of $f(1.2, 1.1, 0.9)$ is then $L(1.2, 1.1, 0.9) = \boxed{1.3}$. (The actual value is ≈ 1.2575.)

- Example: Use a linearization to approximate the change in $f(x,y,z) = e^{x+y}(y+z)^2$ in moving from $(-1,1,1)$ to $(-0.9, 0.9, 1.2)$.
  - First we compute the linearization: we have $f_x = e^{x+y}(y+z)^2$, $f_y = e^{x+y}(y+z)^2 + 2e^{x+y}(y+z)$, and $f_z = 2e^{x+y}(y+z)$, so $\nabla f(-1,1,1) = \langle 4, 8, 4 \rangle$, and then the linearization is $L(x,y,z) = 4 + 4(x+1) + 8(y-1) + 4(z-1)$.
  - We see that the approximate change is then $L(-0.9, 0.9, 1.2) - f(-1,1,1) = 4.4 - 4 = \boxed{0.4}$.
  - For comparison, the actual change is $f(-0.9, 0.9, 1.2) - f(-1,1,1) = e^0 \cdot (2.1)^2 - e^0 \cdot 2^2 = 0.41$. So the approximation here was very good.
  - Note that we could also have estimated this change using a directional derivative: the result is $\nabla f(-1,1,1) \cdot \langle 0.1, -0.1, 0.2 \rangle = 0.4$. This estimate is exactly the same as the one arising from the linearization; this should not be surprising, since the two calculations are ultimately the same.

- In approximating a function by its best linear approximation, we might like to be able to bound how far off our approximations are: after all, an approximation is not very useful if we do not know how good it is!
  - We can give an upper bound on the error using a multivariable version of Taylor's Remainder Theorem, but first we need to discuss Taylor series.

2.4.2 Taylor Series

- There is no reason only to consider linear approximations, aside from the fact that linear functions are easiest: we could just as well ask about how to approximate $f(x, y)$ with a higher-degree polynomial in x and y. This, exactly in analogy with the one-variable case, leads us to Taylor series.
  - We will only write down the formulas for functions of 2 variables. However, all of the results extend naturally to 3 or more variables.

- Definition: If $f(x, y)$ is a function all of whose nth-order partial derivatives at $(x, y) = (a, b)$ exist for every n, then the Taylor series for $f(x)$ at $(x, y) = (a, b)$ is $T(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(x-a)^k (y-b)^{n-k}}{k!(n-k)!} \frac{\partial^n f}{(\partial x)^k (\partial y)^{n-k}}(a, b)$.
  - The series looks like $T(x, y) = f(a, b) + [(x-a)f_x + (y-b)f_y] + \frac{1}{2!}\left[(x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}\right] + \cdots$, where all of the partial derivatives are evaluated at (a, b).
  - At $(a, b) = (0, 0)$ the series is $T(0, 0) = f(0, 0) + [x f_x + y f_y] + \frac{1}{2!}\left[x^2 f_{xx} + 2xy\, f_{xy} + y^2 f_{yy}\right] + \frac{1}{3!}\left[x^3 f_{xxx} + 3x^2 y\, f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right] + \cdots$.
  - It is not entirely clear from the definition, but one can check that for any p and q, $\frac{\partial^{p+q}}{(\partial x)^p (\partial y)^q} f(a, b) = \frac{\partial^{p+q}}{(\partial x)^p (\partial y)^q} T(a, b)$: in other words, the Taylor series has the same partial derivatives as the original function at (a, b).
  - There are, of course, generalizations of the Taylor series to functions of more than two variables. However, the formulas are more cumbersome to write down, so we will not discuss them.

- Example: Find the terms up to degree 3 in the Taylor series for the function $f(x, y) = \sin(x + y)$ near $(0, 0)$.
  - We have $f_x = f_y = \cos(x + y)$, $f_{xx} = f_{xy} = f_{yy} = -\sin(x + y)$, and $f_{xxx} = f_{xxy} = f_{xyy} = f_{yyy} = -\cos(x + y)$.
  - So $f_x(0, 0) = f_y(0, 0) = 1$, $f_{xx}(0, 0) = f_{xy}(0, 0) = f_{yy}(0, 0) = 0$, and $f_{xxx}(0, 0) = f_{xxy}(0, 0) = f_{xyy}(0, 0) = f_{yyy}(0, 0) = -1$.
  - Thus we get $T(x, y) = \boxed{0 + [x + y] + \frac{1}{2!}[0 + 0 + 0] + \frac{1}{3!}\left[-x^3 - 3x^2 y - 3xy^2 - y^3\right] + \cdots}$.
  - Alternatively, we could have used the single-variable Taylor series for sine to obtain this answer: we know that the Taylor expansion of sine is $T(t) = t - \frac{t^3}{3!} + \cdots$. Now we merely set $t = x + y$ to get $T(x, y) = \boxed{(x + y) - \frac{(x + y)^3}{3!} + \cdots}$, which agrees with the partial expansion we found above.

- Definition: We define the degree-d Taylor polynomial of $f(x, y)$ at $(x, y) = (a, b)$ to be the terms in the Taylor series up to degree d in x and y: in other words, the sum $T_d(x, y) = \sum_{n=0}^{d} \sum_{k=0}^{n} \frac{(x-a)^k (y-b)^{n-k}}{k!(n-k)!} \frac{\partial^n f}{(\partial x)^k (\partial y)^{n-k}}(a, b)$.
  - For example, the linear Taylor polynomial of f is $T(x, y) = f(a, b) + [(x-a)f_x(a, b) + (y-b)f_y(a, b)]$, which we recognize is merely another way to write the tangent plane's linear approximation to f at (a, b).
  - As with Taylor series in one variable, the degree-d Taylor polynomial to $f(x, y)$ at $(x, y) = (a, b)$ is the polynomial which has the same derivatives up to dth order as f does at (a, b).

- We have a multivariable version of Taylor's Remainder Theorem, which provides an upper bound on the error from an approximation of a function by its Taylor polynomial:

- Theorem (Taylor's Remainder Theorem, multivariable): If $f(x,y)$ has continuous partial derivatives up to order $d+1$ near (a,b), and if $T_d(x,y)$ is the degree-d Taylor polynomial for $f(x,y)$ at (a,b), then for any point (x,y), we have $|T_k(x,y) - f(x,y)| \le M \cdot \dfrac{(|x-a|+|y-b|)^{k+1}}{(k+1)!}$, where M is a constant such that $|f_\star| \le M$ for every $(d+1)$-order partial derivative $f_\star$ on the segment joining (a,b) to (x,y).
  - The proof of this theorem follows, after some work, by applying the chain rule along with the single-variable Taylor's Theorem to the function $g(t) = f(a+t(x-a), b+t(x-b))$: in other words, by applying Taylor's Theorem to f along the line segment joining (a,b) to (x,y).
- Example: Find the linear polynomial and quadratic polynomial that best approximate $f(x,y) = e^x \cos(y)$ near $(0,0)$. Then give an upper bound on the error from the quadratic approximation in the region where $|x| \le 0.1$ and $|y| \le 0.1$.
  - We have $f_x = f_{xx} = e^x \cos(y)$, $f_y = f_{xy} = -e^x \sin(y)$, and $f_{yy} = -e^x \cos(y)$. We also have $(a,b) = (0,0)$.
  - Then $f(0,0) = 1$, $f_x(0,0) = f_{xx}(0,0) = 1$, $f_y(0,0) = f_{xy}(0,0) = 0$, and $f_{yy}(0,0) = -1$.
  - Hence the best linear approximation is $T_1(x,y) = \boxed{1 + [1\cdot x + 0 \cdot y] = 1 + x}$, and the best quadratic approximation is $T_2(x,y) = 1 + [1 \cdot x + 0 \cdot y] + \dfrac{1}{2!}\left[1 \cdot x^2 + 2 \cdot 0 \cdot xy - 1 \cdot y^2\right] = \boxed{1 + x + \dfrac{x^2}{2} - \dfrac{y^2}{2}}$.
  - For the quadratic error estimate, Taylor's Theorem dictates that $|T_2(x,y) - f(x,y)| \le M \cdot \dfrac{(|x|+|y|)^3}{3!}$, where M is an upper bound on the size of all of the third-order partial derivatives of f in the region where $|x| \le 0.1$ and $|y| \le 0.1$.
  - We can observe by direct calculation (or by observing the pattern) that all of the partial derivatives of f are of the form $\pm e^x \sin(y)$ or $\pm e^x \cos(y)$. Each of these is bounded above by e^x, which on the region where $|x| \le 0.1$, is at most $e^{0.1}$.
  - Thus we can take $M = e^{0.1}$, and since $|x| \le 0.1$ and $|y| \le 0.1$ we obtain the bound $|T_2(x,y) - f(x,y)| \le M \cdot \dfrac{(|x|+|y|)^3}{3!} \le \boxed{e^{0.1} \cdot \dfrac{(0.2)^3}{3!} < 1.2 \cdot \dfrac{(0.008)}{6} = 0.0016}$.
  - Remark: Note that by using the one-variable Taylor's theorem on $e^x = 1 + x + \dfrac{1}{2}x^2 + \cdots$, we can see that $e^x < 1 + 2x$ for $x < 1$, from which we can see that $e^{0.1} < 1.2$.

2.5 Local Extreme Points and Optimization

- Now that we have developed the basic ideas of derivatives for functions of several variables, we can tackle the multivariable version of one of the fundamental applications of differential calculus: that of finding minimum and maximum values of functions.
  - We will primarily discuss functions of two variables, because, as we will discuss, there is a relatively simple criterion for deciding whether a critical point is a minimum or a maximum.
  - Classifying critical points for functions of more than two variables requires some results from linear algebra, so we will not treat the classification problem for functions of more than two variables.

2.5.1 Critical Points, Minima and Maxima, Saddle Points, Classifying Critical Points

- We would first like to determine where a function f can have a minimum or maximum value.
  - We know that the gradient vector ∇f, evaluated at a point P, gives a vector pointing in the direction along which f increases most rapidly at P.
  - If $\nabla f(P) \ne \mathbf{0}$, then f cannot have a local extreme point at P, since f increases along $\nabla f(P)$ and decreases along $-\nabla f(P)$.

- Thus, extreme points can only occur at points where the gradient ∇f is the zero vector (or if it is undefined).

- Definition: A critical point of a function $f(x, y)$ is a point where ∇f is zero or undefined. Equivalently, (x_0, y_0) is a critical point if $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$, or either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ is undefined.
  - By the observations above, a local minimum or maximum of a function can only occur at a critical point.

- Example: Find the critical points of the functions $g(x, y) = x^2 + y^2$ and $h(x, y) = x^2 - y^2$.
  - We have $\nabla g = \langle 2x, 2y \rangle$ and $\nabla h = \langle 2x, -2y \rangle$, which are both defined everywhere.
  - We see immediately that the only solution to $\nabla g = \langle 0, 0 \rangle$ is $(x, y) = (0, 0)$, and the same is true for ∇h.
  - Thus, the origin $\boxed{(0, 0)}$ is the unique critical point for each function.

- Example: Find the critical points of the function $p(x, y) = x^3 + y^3 - 3xy$.
  - We have $\nabla p = \left\langle 3x^2 - 3y,\, 3y^2 - 3x \right\rangle$, which is defined everywhere, so the only critical points will occur when $\nabla p = \langle 0, 0 \rangle$.
  - This gives the two equations $3x^2 - 3y = 0$ and $3y^2 - 3x = 0$, or, equivalently, $x^2 = y$ and $y^2 = x$.
  - Plugging the first equation into the second yields $x^4 = x$: thus, $x^4 - x = 0$, and factoring yields $x(x - 1)(x^2 + x + 1) = 0$.
  - The only real solutions are $x = 0$ (which then gives $y = x^2 = 0$) and $x = 1$ (which then gives $y = x^2 = 1$).
  - Therefore, there are two critical points: $\boxed{(0, 0) \text{ and } (1, 1)}$.

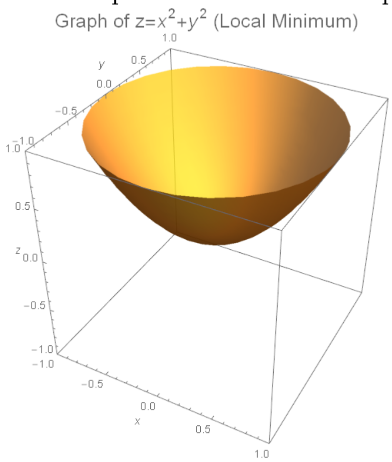
- Example: Find the critical points of the function $q(x, y) = x^2 + 3xy + 2y^2 - 5x - 8y + 4$.
  - We have $\nabla q = \langle 2x + 3y - 5,\, 3x + 4y - 8 \rangle$, which is defined everywhere, so the only critical points will occur when $\nabla q = \langle 0, 0 \rangle$. This yields the equations $2x + 3y - 5 = 0$ and $3x + 4y - 8 = 0$.
  - Setting $2x + 3y - 5 = 0$ and solving for y yields $y = (5 - 2x)/3$. Plugging into the second equation and simplifying eventually gives $x/3 - 4/3 = 0$, so that $x = 4$ and then $y = -1$.
  - Therefore, there is a single critical point $(x, y) = \boxed{(4, -1)}$.

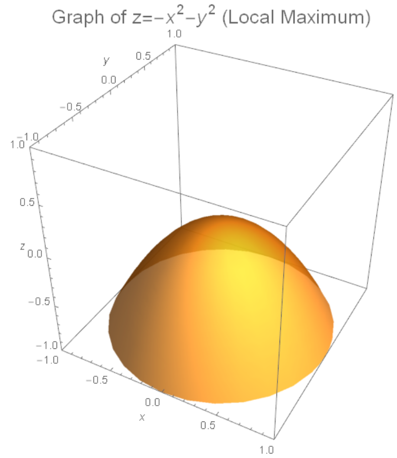
- Example: Find the critical points of the function $f(x, y) = x \sin(y)$.
  - We compute $\nabla f = \langle \sin(y),\, x \cos(y) \rangle$, which is defined everywhere, so the only critical points will occur when $\nabla f = \langle 0, 0 \rangle$.
  - This gives $\sin(y) = 0$ and $x \cos(y) = 0$. The first equation implies $y = \pi k$ for some integer k, and then since $\cos(\pi k) = (-1)^k$ the second equation gives $x = 0$.
  - Thus, the critical points are $(x, y) = \boxed{(0, \pi k) \text{ for any integer } k}$.

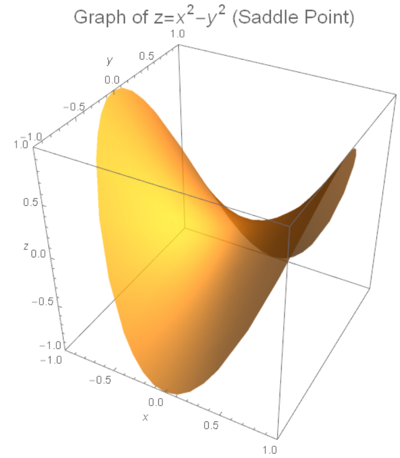
- We have various different types of critical points:

- Definition: A local minimum is a critical point where f is "nearby" always bigger, a local maximum is a critical point where f is "nearby" always smaller, and a saddle point is a critical point where f "nearby" is bigger in some directions and smaller in others.
  - Example: The function $g(x, y) = x^2 + y^2$ has a local minimum at the origin.
  - Example: The function $p(x, y) = -(x^2 + y^2)$ has a local maximum at the origin.
  - Example: The function $h(x, y) = x^2 - y^2$ has a saddle point at the origin, since $h > 0$ along the x-axis (since $h(x, 0) = x^2$) but $h < 0$ along the y-axis (since $h(0, y) = -y^2$).

- Here are plots of the three examples:







- Once we can identify all of a function's critical points, we would like to know whether those points actually are minima or maxima of f. We can determine this using a quantity called the discriminant:

- Definition: The discriminant (also called the Hessian) at a critical point is the value $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$, where each of the second-order partials is evaluated at the critical point.

  - One way to remember the definition of the discriminant is as the determinant of the matrix of the four second-order partials: $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$. (We are implicitly using the fact that $f_{xy} = f_{yx}$.)
  - Example: For $g(x, y) = x^2 + y^2$ we have $g_{xx} = g_{yy} = 2$ and $g_{xy} = 0$ so $D = 4$ at the origin.
  - Example: For $h(x, y) = x^2 - y^2$ we have $h_{xx} = 2$, $h_{yy} = -2$, and $h_{xy} = 0$ so $D = -4$ at the origin.
  - Remark: The reason this value is named "discriminant" can be seen by computing D for the function $p(x, y) = ax^2 + bxy + cy^2$: the result is $D = 4ac - b^2$, which is -1 times the quantity $b^2 - 4ac$, the famous discriminant for the quadratic polynomial $ax^2 + bx + c$. (Recall that the discriminant of $ax^2 + bx + c$ determines how many real roots the polynomial has.)

- Theorem (Second Derivatives Test): Suppose P is a critical point of $f(x, y)$, and let D be the value of the discriminant $f_{xx}f_{yy} - f_{xy}^2$ at P. If $D > 0$ and $f_{xx} > 0$, then the critical point is a minimum. If $D > 0$ and $f_{xx} < 0$, then the critical point is a maximum. If $D < 0$, then the critical point is a saddle point. (If $D = 0$, then the test is inconclusive.)

  - Proof (outline): Assume for simplicity that P is at the origin. Then (by our results on Taylor series), the function $f(x, y) - f(P)$ will be closely approximated by the polynomial $ax^2 + bxy + cy^2$, where $a = \frac{1}{2}f_{xx}$, $b = f_{xy}$, and $c = \frac{1}{2}f_{yy}$. If $D \neq 0$, then the behavior of $f(x, y)$ near the critical point P will be the same as that quadratic polynomial. Completing the square and examining whether the resulting quadratic polynomial has any real roots and whether it opens or downwards yields the test.

- We can combine the above results to yield a procedure for finding and classifying the critical points of a function $f(x, y)$:

  - Step 1: Compute both partial derivatives f_x and f_y.
  - Step 2: Find all points (x, y) where *both* partial derivatives are zero, or where (at least) *one* of the partial derivatives is undefined.
    - It may require some algebraic manipulation to find the solutions: a basic technique is to solve one equation for one of the variables, and then plug the result into the other equation. Another technique is to try to factor one of the equations and then analyze cases.
  - Step 3: At *each* critical point, evaluate $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$ and apply the Second Derivatives Test:

If $D > 0$ and $f_{xx} > 0$: local minimum. If $D > 0$ and $f_{xx} < 0$: local maximum. If $D < 0$: saddle point.

- Example: Verify that $f(x, y) = x^2 + y^2$ has only one critical point, a local minimum at the origin.
  - First, we have $f_x = 2x$ and $f_y = 2y$. Since they are both defined everywhere, we need only find where they are both zero.
  - Setting both partial derivatives equal to zero yields $x = 0$ and $y = 0$, so the only critical point is $(0, 0)$.
  - To classify the critical points, we compute $f_{xx} = 2$, $f_{xy} = 0$, and $f_{yy} = 2$. Then $D = 2 \cdot 2 - 0^2 = 4$.
  - So, by the classification test, since $D > 0$ and $f_{xx} > 0$ at $(0, 0)$, we see that $\boxed{(0,0) \text{ is a local minimum}}$.

- Example: For the function $f(x, y) = 3x^2 + 2y^3 - 6xy$, find the critical points and classify them as minima, maxima, or saddle points.
  - First, we have $f_x = 6x - 6y$ and $f_y = 6y^2 - 6x$. Since they are both defined everywhere, we need only find where they are both zero.
  - Next, we can see that f_x is zero only when $y = x$. Then the equation $f_y = 0$ becomes $6x^2 - 6x = 0$, which by factoring we can see has solutions $x = 0$ or $x = 1$. Since $y = x$, we conclude that $\boxed{(0,0), \text{ and } (1,1) \text{ are critical points}}$.
  - To classify the critical points, we compute $f_{xx} = 6$, $f_{xy} = -6$, and $f_{yy} = 12y$. Then $D(0, 0) = 6 \cdot 0 - (-6)^2 < 0$ and $D(1, 1) = 6 \cdot 12 - (-6)^2 > 0$.
  - So, by the classification test, $\boxed{(0,0) \text{ is a saddle point}}$ and $\boxed{(1,1) \text{ is a local minimum}}$.

- Example: For the function $g(x, y) = x^3y - 3xy^3 + 8y$, find the critical points and classify them as minima, maxima, or saddle points.
  - First, we have $g_x = 3x^2y - 3y^3$ and $g_y = x^3 - 9xy^2 + 8$. Since they are both defined everywhere, we need only find where they are both zero.
  - Setting both partial derivatives equal to zero. Since $g_x = 3y(x^2 - y^2) = 3y(x + y)(x - y)$, we see that $g_x = 0$ precisely when $y = 0$ or $y = x$ or $y = -x$.
  - If $y = 0$, then $g_y = 0$ implies $x^3 + 8 = 0$, so that $x = -2$. This yields the point $(x, y) = (-2, 0)$.
  - If $y = x$, then $g_y = 0$ implies $-8x^3 + 8 = 0$, so that $x = 1$. This yields the point $(x, y) = (1, 1)$.
  - If $y = -x$, then $g_y = 0$ implies $-8x^3 + 8 = 0$, so that $x = 1$. This yields the point $(x, y) = (1, -1)$.
  - To summarize, we see that $\boxed{(-2,0),\ (1,1), \text{ and } (1,-1) \text{ are critical points}}$.
  - To classify the critical points, we compute $g_{xx} = 6xy$, $g_{xy} = 3x^2 - 9y^2$, and $g_{yy} = -18xy$.
  - Then $D(-2, 0) = 0 \cdot 0 - (12)^2 < 0$, $D(1, 1) = 6 \cdot (-18) - (-6)^2 < 0$, and $D(1, -1) = (-6) \cdot (18) - (-6)^2 < 0$.
  - So, by the classification test, $\boxed{(-2,0),\ (1,1), \text{ and } (1,-1) \text{ are all saddle points}}$.

- By searching for critical points, we can solve optimization problems:
- Example: Find the minimum value of the function $h(x, y) = x + 2y^4 - \ln(x^4y^8)$, for x and y positive.
  - To solve this problem, we will search for all critical points of $h(x, y)$ that are minima.
  - First, we have $h_x = 1 - \dfrac{4x^3y^8}{x^4y^8} = 1 - \dfrac{4}{x}$ and $h_y = 8y^3 - \dfrac{8x^4y^7}{x^4y^8} = 8y^3 - \dfrac{8}{y}$. Both partial derivatives are defined everywhere in the given domain.
  - We see that $h_x = 0$ only when $x = 4$, and also that $h_y = 0$ is equivalent to $\dfrac{8}{y}(y^4 - 1) = 0$, which holds for $y = \pm 1$. Since we only want $y > 0$, there is a unique critical point: $(4, 1)$.
  - Next, we compute $h_{xx} = \dfrac{4}{x^2}$, $g_{xy} = 0$, and $g_{yy} = 24y^2 + \dfrac{8}{y^2}$. Then $D(4, 1) = \dfrac{1}{4} \cdot 32 - 0^2 > 0$.

- Thus, there is a unique critical point, and it is a minimum. Therefore, we conclude that the function has its minimum at $(4,1)$, and the minimum value is $h(4,1) = \boxed{6 - \ln(4^4)}$.

- Example: Find the minimum distance between a point on the plane $x + y + z = 1$ and the point $(2, -1, -2)$.
  - The distance from the point (x, y, z) to $(2, -1, 2)$ is $d = \sqrt{(x-2)^2 + (y+1)^2 + (z+2)^2}$. Since $x+y+z = 1$ on the plane, we can view this as a function of x and y only: $d(x, y) = \sqrt{(x-2)^2 + (y+1)^2 + (3-x-y)^2}$.
  - We could minimize $d(x, y)$ by finding its critical points and searching for a minimum, but it will be much easier to find the minimum value of the squared distance $f(x, y) = d(x, y)^2 = (x-2)^2+(y+1)^2+(3-x-y)^2$.
  - We compute $f_x = 2(x-2) - 2(3-x-y) = 4x+2y-10$ and $f_y = 2(y+1) - 2(3-x-y) = 2x+4y-4$. Both partial derivatives are defined everywhere, so we need only find where they are both zero.
  - Setting $f_x = 0$ and solving for y yields $y = 5-2x$, and then plugging this into $f_y = 0$ yields $2x + 4(5-2x) - 4 = 0$, so that $-6x + 16 = 0$. Thus, $x = 8/3$ and then $y = -1/3$.
  - Furthermore, we have $f_{xx} = 4$, $f_{xy} = 2$, and $f_{yy} = 4$, so that $D = f_{xx}f_{yy} - f_{xy}^2 = 12 > 0$. Thus, the point $(x, y) = (8/3, -1/3)$ is a local minimum.
  - Thus, there is a unique critical point, and it is a minimum. We conclude that the distance function has its minimum at $(8/3, -1/3)$, so the minimum distance is $d(8/3, -1/3) = \sqrt{(2/3)^2 + (2/3)^2 + (2/3)^2} = \boxed{2/\sqrt{3}}$.
  - Notice that this calculation agrees with the result of using the point-to-plane distance formula (as it should!): the formula gives the minimal distance as $d = |2-1-2-1| / \sqrt{1^2+1^2+1^2} = 2/\sqrt{3}$.

2.5.2 Optimization of a Function on a Region

- If we instead want to find the absolute minimum or maximum of a function $f(x, y)$ on a region of the plane (rather than on the entire plane) we must also analyze the function's behavior on the boundary of the region, because the boundary could contain the minimum or maximum.
  - Example: The extreme values of $f(x, y) = x^2 - y^2$ on the square $0 \le x \le 1$, $0 \le y \le 1$ occur at two of the "corner points": the minimum is -1 occurring at $(0, 1)$, and the maximum $+1$ occurring at $(1, 0)$. We can see that these two points are actually the minimum and maximum on this region without calculus: since squares of real numbers are always nonnegative, on the region in question we have $-1 \le -y^2 \le x^2 - y^2 \le x^2 \le 1$.

- Unfortunately, unlike the case of a function of one variable where the boundary of an interval $[a, b]$ is very simple (namely, the two values $x = a$ and $x = b$), the boundary of a region in the plane or in higher-dimensional space can be rather complicated.
  - One approach is to find a parametrization $(x(t), y(t))$ of the boundary of the region. (This may require breaking the boundary into several pieces, depending on the shape of the region.)
  - Then, by plugging the parametrization of the boundary curve into the function, we obtain a function $f(x(t), y(t))$ of the single variable t, which we can then analyze to determine the behavior of the function on the boundary.
  - Another approach, detailed in the next section, is to use Lagrange multipliers.

- To find the absolute minimum and maximum values of a function on a given region R, follow these steps:
  - Step 1: Find all of the critical points of f that lie inside the region R.
  - Step 2: Parametrize the boundary of the region R (separating into several components if necessary) as $x = x(t)$ and $y = y(t)$, then plug in the parametrization to obtain a function of t, $f(x(t), y(t))$. Then search for "boundary-critical" points, where the t-derivative $\frac{d}{dt}$ of $f(x(t), y(t))$ is zero. Also include endpoints, if the boundary components have them.
    - A line segment from (a, b) to (c, d) can be parametrized by $x(t) = a + t(c-a)$, $y(t) = b + t(d-b)$, for $0 \le t \le 1$.

    * A circle of radius r with center (h,k) can be parametrized by $x(t) = h + r\cos(t)$, $y(t) = k + r\sin(t)$, for $0 \le t \le 2\pi$.

  - Step 3: Plug the full list of critical and boundary-critical points into f, and find the largest and smallest values.

- Example: Find the absolute minimum and maximum of $f(x,y) = x^3 + 6xy - y^3$ on the triangle with vertices $(0,0)$, $(4,0)$, and $(0,-4)$.

  - First, we find the critical points: we have $f_x = 3x^2 + 6y$ and $f_y = -3y^2 + 6x$. Solving $f_y = 0$ yields $x = y^2/2$ and then plugging into $f_x = 0$ gives $y^4/4 + 2y = 0$ so that $y(y^3 + 8) = 0$: thus, we see that $(0,0)$ and $(2,-2)$ are critical points.
  - Next, we analyze the boundary of the region. Here, the boundary has 3 components.
    * Component #1, joining $(0,0)$ to $(4,0)$: This component is parametrized by $x = t$, $y = 0$ for $0 \le t \le 4$. On this component we have $f(t,0) = t^3$, which has a critical point only at $t = 0$, which corresponds to $(x,y) = (0,0)$. Also add the boundary point $(4,0)$.
    * Component #2, joining $(0,-4)$ to $(4,0)$: This component is parametrized by $x = t$, $y = t - 4$ for $0 \le t \le 4$. On this component we have $f(t, t-4) = 18t^2 - 72t + 64$, which has a critical point for $t = 2$, corresponding to $(x,y) = (2,-2)$. Also add the boundary points $(4,0)$ and $(0,-4)$.
    * Component #3, joining $(0,0)$ to $(0,-4)$: This component is parametrized by $x = 0$, $y = -t$ for $0 \le t \le 4$. On this component we have $f(0,t) = t^3$, which has a critical point for $t = 0$, corresponding to $(x,y) = (0,0)$. Also add the boundary point $(0,-4)$.
  - Our full list of points to analyze is $(0,0)$, $(4,0)$, $(0,-4)$, and $(2,-2)$. We compute $f(0,0) = 0$, $f(4,0) = 64$, $f(0,-4) = 64$, $f(2,-2) = -8$, and so we see that $\boxed{\text{maximum is 64}}$ and the $\boxed{\text{minimum is -8}}$.

- Example: Find the absolute maximum and minimum of $f(x,y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \le 4$.

  - First, we find the critical points: we have $f_x = 2x$ and $f_y = -2y$. Clearly both are zero only at $(x,y) = (0,0)$, so $(0,0)$ is the only critical point.
  - Next, we analyze the boundary of the region. The boundary is the circle $x^2 + y^2 = 4$, which is a single curve parametrized by $x = 2\cos(t)$, $y = 2\sin(t)$ for $0 \le t \le 2\pi$.
    * On the boundary, therefore, we have $f(t) = f(2\cos(t), 2\sin(t)) = 4\cos^2 t - 4\sin^2 t$.
    * Taking the derivative yields $\dfrac{df}{dt} = 4\left[2\cos(t)\cdot(-\sin(t)) - 2\sin(t)\cdot\cos(t)\right] = -8\cos(t)\cdot\sin(t)$.
    * The derivative is equal to zero when $\cos(t) = 0$ or when $\sin(t) = 0$. For $0 \le t \le 2\pi$ this gives the possible values $t = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi$, yielding $(x,y) = (2,0), (0,2), (-2,0), (0,-2)$, and $(2,0)$ [again].
    * Note: We could have saved a small amount of effort by observing that $f(t) = 4\cos^2 t - 4\sin^2 t$ is also equal to $4\cos(2t)$.
  - Our full list of points to analyze is $(0,0)$, $(2,0)$, $(0,2)$, $(-2,0)$, and $(0,-2)$. We have $f(0,0) = 0$, $f(2,0) = 4$, $f(0,2) = -4$, $f(-2,0) = 4$, and $f(0,-2) = -4$. The $\boxed{\text{maximum is 4}}$ and the $\boxed{\text{minimum is -4}}$.

- Example: Find the absolute maximum and minimum of $f(x,y) = xy - 3x$ on the region with $x^2 \le y \le 9$.

  - First, we find the critical points: since $f_x = y - 3$ and $f_y = x$, there is a single critical point $\boxed{(0,3)}$.
  - Next, we analyze the boundary of the region, which (as a quick sketch reveals) has 2 components.
    * Component #1, a line segment from $(-3,9)$ to $(3,9)$: This component is parametrized by $x = t$, $y = 9$ for $-3 \le t \le 3$. On this component we have $f(t,9) = 6t$, which has no critical point. We only have boundary points $\boxed{(-3,9),\ (3,9)}$.
    * Component #2, a parabolic arc parametrized by $x = t$, $y = t^2$ for $-3 \le t \le 3$. On this component we have $f(t,t^2) = t^3 - 3t$, which has critical points at $t = \pm 1$, corresponding to $(x,y) = \boxed{(-1,1),\ (1,1)}$. The boundary points $(-3,9)$, $(3,9)$ are already listed above.
  - Our full list of points to analyze is $(0,3)$, $(-3,9)$, $(3,9)$, $(-1,1)$, $(1,1)$. We have $f(0,3) = 0$, $f(-3,9) = -18$, $f(3,9) = 18$, $f(-1,1) = 2$, and $f(1,1) = -2$. The $\boxed{\text{maximum is 18}}$ and the $\boxed{\text{minimum is -18}}$.

2.6 Lagrange Multipliers and Constrained Optimization

- Many types of applied optimization problems are not of the form "given a function, maximize it on a region", but rather of the form "given a function, maximize it subject to some additional constraints".
  - Example: Maximize the volume $V = \pi r^2 h$ of a cylindrical can given that its surface area $SA = 2\pi r^2 + 2\pi r h$ is $150\pi\,\text{cm}^2$.
- The most natural way to (try to) solve such a problem is to eliminate the constraints by solving for one of the variables in terms of the others and then reducing the problem to something without a constraint. Then we are able to perform the usual procedure of evaluating the derivative (or derivatives), setting them equal to zero, and looking among the resulting critical points for the desired extreme point.
  - In the example above, we would use the surface area constraint $150\pi\,\text{cm}^2 = 2\pi r^2 + 2\pi r h$ to solve for h in terms of r, obtaining $h = \dfrac{150\pi - 2\pi r^2}{2\pi r} = \dfrac{75 - r^2}{r}$, and then plug in to the volume formula to write it as a function of r alone: this gives $V(r) = \pi r^2 \cdot \dfrac{75 - r^2}{r} = 75\pi r - \pi r^3$.
  - Then $dV/dr = 75\pi - 3\pi r^2$, so by (setting equal to zero) we see the critical points occur for $r = \pm 5$.
  - Since we are interested in positive r, we can do a little bit more checking to conclude that the can's volume is indeed maximized at the critical point, so the radius is $r = 5\,\text{cm}$, the height is $h = 10\,\text{cm}$, and the resulting volume is $V = 250\pi\text{cm}^3$.
- Using the technique of Lagrange multipliers, however, we can perform a constrained optimization without having to solve the constraint equations. This technique is especially useful when the constraints are difficult or impossible to solve explicitly.
- Method (Lagrange multipliers, 1 constraint): To find the extreme values of $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, assuming $\nabla g \neq \mathbf{0}$ subject to the constraint, it is sufficient to solve the system of four variables x, y, z, λ given by $\nabla f = \lambda \nabla g$ and $g(x, y, z) = c$, and then search among the resulting triples (x, y, z) to find the minimum and maximum.
  - The value λ is called a Lagrange multiplier, which is where the name of the procedure comes from.
  - Another way of interpreting the hypothesis of the method of Lagrange multipliers is to observe that, if one defines the "Lagrange function" to be $L(x, y, z, \lambda) = f(x, y, z) - \lambda \cdot [g(x, y, z) - c]$, then the minimum and maximum of $f(x, y, z)$ subject to $g(x, y, z) = c$ all occur at critical points of L.
  - If we have fewer or more variables (but still one constraint), the setup is the same: $\nabla f = \lambda \nabla g$ and $g = c$.
  - Proof: Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ be any parametric curve on the surface $g(x, y, z) = c$ that passes through a local extreme point of f at $t = 0$.
  - Applying the chain rule to f and g yields $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\, x'(t) + \dfrac{\partial f}{\partial y}\, y'(t) + \dfrac{\partial f}{\partial z}\, z'(t) = \nabla f \cdot \mathbf{r}'(t)$ and $\dfrac{\partial g}{\partial t} = \dfrac{\partial g}{\partial x}\, x'(t) + \dfrac{\partial g}{\partial y}\, y'(t) + \dfrac{\partial g}{\partial z}\, z'(t) = \nabla g \cdot \mathbf{r}'(t)$.
  - Since g is a constant function on the surface on the surface $g(x, y, z) = c$ and the curve $\mathbf{r}(t)$ lies on that surface, we have $\partial g / \partial t = 0$ for all t.
  - Also, since f has a local extreme point at $t = 0$, we have $\partial f / \partial t = 0$ at $t = 0$.
  - Thus at $t = 0$ we have $\nabla f(0) \cdot \mathbf{r}'(0) = 0$ and $\nabla g(0) \cdot \mathbf{r}'(0) = 0$.
  - This holds for every parametric curve $\mathbf{r}(t)$ passing through the local extreme point. One can show using a geometric or algebraic argument that this requires that ∇f and ∇g be parallel at $t = 0$.
  - But, when $\nabla g \neq \mathbf{0}$, this is precisely the statement that, at the extreme point, $\nabla f = \lambda \nabla g$. We are also restricted to the surface $g(x, y, z) = c$, so this equation holds as well.
- Here is the intuitive idea behind the method of Lagrange multipliers:
  - Imagine we are walking around the level set $g(x, y, z) = c$, and consider what the contours of $f(x, y, z)$ are doing as we move around.

- In general the contours of f and g will be different, and they will cross one another.
- But if we are at a point where f is maximized, then if we walk around nearby that maximum, we will see only contours of f with a smaller value than the maximum.
- Thus, at that maximum, the contour $g(x, y, z) = c$ is tangent to the contour of f.
- Since the gradient is orthogonal to (any) tangent curve, this is equivalent to saying that, at a maximum, the gradient vector of f is parallel to the gradient vector of g, or in other words, there exists a scalar λ for which $\nabla f = \lambda \nabla g$.

• Example: Find the maximum and minimum values of $f(x, y) = 2x+3y$ subject to the constraint $x^2+4y^2 = 100$.

- We use Lagrange multipliers: we have $g = x^2 + 4y^2$, so $\nabla f = \langle 2,\, 3\rangle$ and $\nabla g = \langle 2x,\, 8y\rangle$.
- Thus we have the system $2 = 2x\lambda$, $3 = 8y\lambda$, and $x^2 + 4y^2 = 100$.
- Solving the first two equations gives $x = \dfrac{1}{\lambda}$ and $y = \dfrac{3}{8\lambda}$. Then plugging in to the third equation yields $\left(\dfrac{1}{\lambda}\right)^2 + 4\left(\dfrac{3}{8\lambda}\right)^2 = 100$, so that $\dfrac{1}{\lambda^2} + \dfrac{9}{16\lambda^2} = 100$. Multiplying both sides by $16\lambda^2$ yields $25 = 100(16\lambda^2)$, so that $\lambda^2 = 1/64$ hence $\lambda = \pm 1/8$.
- Thus, we obtain the two points $(x, y) = (8, 3)$ and $(-8, -3)$.
- Since $f(8, 3) = 25$ and $f(-8, -3) = -25$, the maximum is $\boxed{f(8, 3) = 25}$ and the minimum is $\boxed{f(-8, -3) = -25}$.

• Example: Find the maximum and minimum values of $f(x, y, z) = x + 2y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 9$.

- We use Lagrange multipliers: we have $g = x^2 + y^2 + z^2$, so $\nabla f = \langle 1, 2, 2\rangle$ and $\nabla g = \langle 2x, 2y, 2z\rangle$.
- Thus we have the system $1 = 2x\lambda$, $2 = 2y\lambda$, $2 = 2z\lambda$, and $x^2 + y^2 + z^2 = 9$.
- Solving the first three equations gives $x = 1/(2\lambda)$, $y = 1/\lambda$, $z = 1/\lambda$; plugging in to the last equation yields $[1/(2\lambda)]^2 + [1/\lambda]^2 + [1/\lambda]^2 = 9$, so that $\frac{9}{4}/\lambda^2 = 9$ and thus $\lambda = \pm 1/2$.
- This gives the two points $(x, y, z) = (1, 2, 2)$ and $(-1, -2, -2)$.
- Since $f(1, 2, 2) = 9$ and $f(-1, -2, -2) = -9$, the maximum is $\boxed{f(1, 2, 2) = 9}$ and the minimum is $\boxed{f(-1, -2, -2) = -9}$.

• Example: Maximize the volume $V = \pi r^2 h$ of a cylindrical can given that its surface area $SA = 2\pi r^2 + 2\pi rh$ is $150\pi\,\mathrm{cm}^2$.

- We use Lagrange multipliers: we have $f(r, h) = \pi r^2 h$ and $g(r, h) = 2\pi r^2+2\pi rh$, so $\nabla f = \left\langle 2\pi rh, \pi r^2\right\rangle$ and $\nabla g = \langle 4\pi r + 2\pi h, 2\pi r\rangle$, yielding the system $2\pi rh = (4\pi r+2\pi h)\lambda$, $\pi r^2 = (2\pi r)\lambda$, and $2\pi r^2+2\pi rh = 150\pi$.
- We clearly cannot have $r = 0$ since that contradicts the third equation, so we can assume $r \neq 0$.
- Solving the second equation yields $\lambda = r/2$. Plugging into the first equation (and cancelling the πs) yields $2rh = (4r + 2h) \cdot r/2$, so dividing by r yields $2h = 2r + h$, so that $h = 2r$.
- Finally, plugging in $h = 2r$ to the third equation (after cancelling the factors of π) yields $2r^2 + 4r^2 = 150$, so that $r^2 = 25$ and $r = \pm 5$.
- The two candidate points are $(r, h) = (5, 10)$ and $(-5, -10)$; since we only want positive values we are left only with $(5, 10)$, which by the physical setup of the problem must be the maximum.
- Therefore, the maximum volume is $\boxed{f(5, 10) = 250\pi\mathrm{cm}^3}$.

• In problems involving optimization on a region, if the boundary of the region can be described as an implicit curve or surface, we can use Lagrange multipliers (rather than a parametrization) to identify any potential boundary-critical points:

• Example: Find the absolute maximum and minimum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

- First, we find the critical points: we have $f_x = 2x$ and $f_y = -2y$. Clearly both are zero only at $(x, y) = (0, 0)$, so $(0, 0)$ is the only critical point.
- Next, we analyze the boundary of the region. The boundary is the circle $x^2 + y^2 = 4$, which we can view as the constraint $g(x, y) = 4$ where $g(x, y) = x^2 + y^2$.
- Now we use Lagrange multipliers: we have $\nabla f = \langle 2x, -2y \rangle$ and $\nabla g = \langle 2x, 2y \rangle$, yielding the system $2x = 2x\lambda$, $-2y = 2y\lambda$, and $x^2 + y^2 = 4$.
- The first equation yields $x = 0$ or $\lambda = 1$. If $x = 0$ then the third equation yields $y^2 = 4$ so that $y = \pm 2$, and then the second equation is satisfied for $\lambda = -1$: this yields two points $(x, y) = (0, 2)$ and $(0, -2)$.
- Otherwise, if $\lambda = 1$ then the second equation yields $y = 0$, and then the third equation gives $x^2 = 4$ so that $x = \pm 2$: this yields two points $(x, y) = (2, 0)$ and $(-2, 0)$.
- Our full list of points to analyze is $(0, 0)$, $(2, 0)$, $(0, 2)$, $(-2, 0)$, and $(0, -2)$. We have $f(0, 0) = 0$, $f(2, 0) = 4$, $f(0, 2) = -4$, $f(-2, 0) = 4$, and $f(0, -2) = -4$. The $\boxed{\text{maximum is 4}}$ and the $\boxed{\text{minimum is -4}}$.

- Example: Find the absolute maximum and minimum of $f(x, y) = x^2 + yz$ on the interior of the ellipsoid $x^2 + 4y^2 + 9z^2 \leq 72$.
  - First, we find the critical points: we have $f_x = 2x$, $f_y = z$, $f_z = y$. Clearly all three are zero only at $(x, y, z) = (0, 0, 0)$, so $(0, 0, 0)$ is the only critical point.
  - Next, we analyze the boundary of the region. The boundary is the ellipsoid $x^2 + 4y^2 + 9z^2 = 72$, which we can view as the constraint $g(x, y) = 72$ where $g(x, y) = x^2 + 4y^2 + 9z^2$.
  - Now we use Lagrange multipliers: we have $\nabla f = \langle 2x, z, y \rangle$ and $\nabla g = \langle 2x, 8y, 18z \rangle$, yielding the system $2x = 2x\lambda$, $z = 8y\lambda$, $y = 18z\lambda$, and $x^2 + 4y^2 + 9z^2 = 1$.
  - Plugging the second equation into the third yields $y = 144\lambda^2 y$ so that $y = 0$ or $\lambda^2 = 1/144$.
  - If $y = 0$ then the second equation would give $z = 0$, and then the fourth equation would become $x^2 = 1$ so that $x = \pm 1$. The first equation then reduces to 1, which is consistent. Thus we obtain two points: $(x, y, z) = (1, 0, 0)$ and $(-1, 0, 0)$.
  - Otherwise, $y \neq 0$ and so $\lambda^2 = 1/144$. Then, because $\lambda \neq 1$, the first equation requires $x = 0$. The system then reduces to $z = \pm\frac{2}{3}y$ and $4y^2 + 9z^2 = 72$, so plugging in yields $4y^2 + 4y^2 = 72$ and thus $y^2 = 9$ so that $y = \pm 3$. Thus we obtain four points: $(x, y, z) = (0, 3, 2)$, $(0, 3, -2)$, $(0, -3, 2)$, and $(0, -3, -2)$.
  - Our full list of points to analyze is $(0, 0, 0)$, $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 3, 2)$, $(0, 3, -2)$, $(0, -3, 2)$, and $(0, -3, -2)$. We have $f(0, 0, 0) = 0$, $f(\pm 1, 0, 0) = 1$, $f(0, 3, 2) = f(0, -3, -2) = 6$, $f(0, 3, -2) = f(0, -3, 2) = -6$. The $\boxed{\text{maximum is 6}}$ and the $\boxed{\text{minimum is -6}}$.
- For completeness we also mention that there is an analogous procedure for a problem with two constraints:
- Method (Lagrange Multipliers, 2 constraints): To find the extreme values of $f(x, y, z)$ subject to a pair of constraints $g(x, y, z) = c$ and $h(x, y, z) = d$, with $\nabla g \neq \mathbf{0}$ and $\nabla h \neq \mathbf{0}$, it is sufficient to solve the system of five variables x, y, z, λ, μ given by $\nabla f = \lambda \nabla g + \mu \nabla h$, $g(x, y, z) = c$, and $h(x, y, z) = d$, and then search among the resulting triples (x, y, z) to find the minimum and maximum.
  - The method also works with more than three variables, and has a natural generalization to more than two constraints. (It is fairly rare to encounter systems with more than two constraints.)

Well, you're at the end of my handout. Hope it was helpful.