Contents

9 Introduction to Differential Equations

- Differential equations crop in every area of anything that can be described mathematically. In every branch of science, from physics to chemistry to biology, as well as other fields such as engineering, economics, and demography, virtually any interesting kind of process is modeled by a differential equation or a system of differential equations.

- Morally, the reason for this is that most anything interesting involves change of some kind, and thus rates of change – in the guise of a growth rate for a population, or the velocity and acceleration of a physical object, or the diffusion rates of molecules involved in a reaction, or rates of quantities in economic processes.

- In general, a differential equation is merely an equation involving a derivative (or several derivatives) of a function or functions.

    - <u>Examples</u>: $y' + y = 0$, or $y'' + 2y' + y = 3x^2$, or $f'' \cdot f = (f')^2$, or $f' + g' = x^3$.

    - "Most" differential equations are difficult if not impossible to find exact solutions to, in the same way that "most" random integrals or infinite series are hard to evaluate exactly.

    - In this course we will only cover how to solve a few basic types of equations: (first-order) separable equations, first-order linear equations, and second-order linear equations with constant coefficients.

- Sometimes we will be looking for every function which satisfies some particular equation: e.g., for all functions such that $y' + y = 0$. Other times we will be looking for a particular function, subject to some additional conditions – e.g., a function y such that $y' + y = 0$ and $y(0) = 10$. We will discuss both types of problems.

9.1 Terminology

- If a differential equation involves functions of only a single variable (i.e., if y is a function only of x) then it is called an <u>ordinary differential equation</u> (or <u>ODE</u>).

    - We will only talk about ODEs in this course, since we don't know how to differentiate functions of more than one variable.

    - For completeness, differential equations involving functions of several variables are called <u>partial differential equations</u>, or PDEs. (Derivatives of functions of more than one variable are called partial derivatives, hence the name.)

- The <u>standard form</u> of a differential equation is when it is written with all terms involving y or higher derivatives on one side, and functions of the variable on the other side.

    ◦ <u>Example</u>: The equation $y'' + y' + y = 0$ is in standard form.
    ◦ <u>Example</u>: The equation $y' = 3x^2 - xy$ is not in standard form.

- An equation is <u>homogeneous</u> if, when it is put into standard form, the x-side is zero. An equation is <u>nonhomogeneous</u> otherwise.

    ◦ <u>Example</u>: The equation $y'' + y' + y = 0$ is homogeneous.
    ◦ <u>Example</u>: The equation $y' + xy = 3x^2$ is nonhomogeneous.

- An <u>nth order differential equation</u> is an equation in which the highest derivative is the nth derivative.

    ◦ <u>Example</u>: The equations $y' + xy = 3x^2$ and $y' \cdot y = 2$ are first-order.
    ◦ <u>Example</u>: The equation $y'' + y' + y = 0$ is second-order.

- A differential equation is <u>linear</u> if it is linear in the y terms. In other words, if there are no terms like y^2, or $(y')^3$, or $y \cdot y'$.

    ◦ <u>Example</u>: The equations $y' + xy = 3x^2$ and $y'' + y' + y = 0$ are linear.
    ◦ <u>Example</u>: The equation $y' \cdot y = 3x^2$ is not linear.

- We say a linear differential equation has <u>constant coefficients</u> if the coefficients of y, y', y'', ... are all constants.

    ◦ <u>Example</u>: The equation $y'' + y' + y = 0$ has constant coefficients.
    ◦ <u>Example</u>: The equation $y' + xy = 3x^2$ does not have constant coefficients.

- <u>Theorem</u>: If y_0 and y_1 both satisfy the same linear homogeneous differential equation (of any order), then $C_0 \cdot y_0 + C_1 \cdot y_1$ will also satisfy the equation, for any constants C_0 and C_1.

    ◦ This is just a formal check. For example, if the differential equation is $y'' + y = 0$, then $[C_0 y_0 + C_1 y_1]'' + [C_0 y_0 + C_1 y_1] = C_0[y_0'' + y_0] + C_1[y_1'' + y_1] = C_0[0] + C_1[0] = 0$.

9.2   Some Motivating Applications

- <u>Simple motivating example</u>: A population (unrestricted by space or resources) tends to grow at a rate proportional to its size. [Reason: imagine each male pairing off with a female and having a fixed number of offspring each year.]

    ◦ In symbols, this means that $\dfrac{dP}{dt} = k \cdot P$, where $P(t)$ is the population at time t and k is the growth rate. This is a homogeneous first-order linear differential equation with constant coefficients.
    ◦ It's not hard to see that one population model that works is $P(t) = e^{k \cdot t}$ – hence, "exponential growth".

- <u>More complicated example</u>: The Happy Sunshine Valley is home to Cute Mice and Adorable Kittens. The Cute Mice grow at a rate proportional to their population, minus the number of Mice that are eaten by their predators, the Kittens. The population of Adorable Kittens grows proportional to the number of mice (since they have to catch Mice to survive and reproduce).

    ◦ In symbols this means $\dfrac{dM}{dt} = k_1 \cdot M - k_2 \cdot K$, and $\dfrac{dK}{dt} = k_3 \cdot M$, where $M(t)$ and $K(t)$ are the populations of Mice and Kittens, and k_1, k_2, k_3 are some constants.
    ◦ Now it's a lot harder to see what a solution to this system could be. (We won't explicitly learn how to solve a system like this, but it can be converted to a single second-order linear equation, which can then be solved using the methods of this course.)

○ The conditions here are not particularly unnatural for a simple predator-prey system. But in general, there could be non-linear terms too – perhaps when two Kittens meet, they fight with each other and cause injury, which might change the equation to $\dfrac{dK}{dt} = k_3 \cdot M - k_4 \cdot K^2$.

○ Now imagine trying to model even a 'small' ecosystem with 5 species, each of which interacts with all of the others.

• Higher-order example: A simple pendulum consists of a weight suspended on a string, with gravity the only force acting on the weight. If θ is the angle the pendulum's string makes with a vertical line, then horizontal force on the weight toward the vertical is proportional to $\sin(\theta)$.

○ In symbols, this means that $\dfrac{d^2\theta}{dt^2} = -k \cdot \sin(\theta)$. This is a non-linear second-order differential equation.

○ This equation cannot be solved exactly for the function $\theta(t)$. However, a reasonably good approximation can be found by using $\sin(\theta) \approx \theta$.

9.3   First-Order: Existence-Uniqueness Theorem

• Before we discuss how to solve various classes of first-order differential equations, we would like to know when (in general) we can say that a first-order differential solution has an equation, and when we can say that solution is unique.

• Theorem (Existence-Uniqueness): The initial value problem $y' = f(x, y)$ with $y(a) = b$ has <u>at least</u> one solution (on some interval containing a) if the function f is continuous on a rectangle containing (a, b). The IVP has <u>exactly one</u> solution (on some interval containing a) if the partial derivative $\dfrac{\partial f}{\partial y}$ is continuous on a rectangle containing (a, b).

○ Note: The partial derivative $\dfrac{\partial f}{\partial y}$ is obtained by treating x as a constant in the definition of f, and differentiating with respect to y. If $f(x, y) = x^3 y^2 + e^{xy}$, for example, then $\dfrac{\partial f}{\partial y} = 2x^3 y + x\, e^{xy}$.

○ The proof of the theorem is fairly difficult. The general idea of one proof of the theorem is to construct a sequence of functions (defined on some small interval around a), such that taking the limit of the sequence yields a solution to the differential equation.

○ The continuity of f ensures that the sequence will converge; one way of doing this is to use the continuity of f to show that the functions far out in the sequence eventually become very close together.

○ The continuity of the partial derivative $\dfrac{\partial f}{\partial y}$ ensures that the solution function is unique; one way of doing this is to use the continuity of $\dfrac{\partial f}{\partial y}$ to show that the integral of the absolute value of the differences of two solutions is zero on an interval containing a.

• Example: Determine the initial conditions $y(a) = b$ for which the differential equation $y' = e^y + xy$ is guaranteed to have a solution, and where it is guaranteed to have a unique solution.

○ All initial conditions lead to a solution, because $f(x, y) = e^y + xy$ is continuous everywhere.

○ In fact, $\boxed{\text{all initial conditions lead to a unique solution}}$, because the partial derivative $f_y(x, y) = e^y + x$ is also continuous everywhere.

○ It is, in fact, not possible to solve this equation explicitly using any of the techniques we will learn. Nonetheless, the theorem guarantees that it has a unique solution!

• Example: Determine the initial conditions $y(a) = b$ for which the differential equation $y' = y^{2/3}$ is guaranteed to have a solution, and where it is guaranteed to have a unique solution.

○ $\boxed{\text{All initial conditions lead to a solution}}$, because since $f(x, y) = y^{2/3}$ is continuous everywhere.

○ However, the partial derivative $f_y = \dfrac{2}{3}y^{-1/3}$ is not continuous near $y = 0$, and so the solution is $\boxed{\text{not guaranteed to be unique around } (a,0)}$ for any a, but $\boxed{\text{unique otherwise}}$.

○ In fact, we can even write down two different solutions to the IVP $y' = y^{1/3}$ with $y(0) = 0$: namely, the constant function $y = 0$ and the function $y = \dfrac{1}{27}x^3$. (They both satisfy the equation and take the value zero at $x = 0$, but are not the same function.)

- Example: Determine the initial conditions $y(a) = b$ for which the differential equation $y' = \sqrt{y-x}$ is guaranteed to have a (real-valued) solution, and where it is guaranteed to have a unique solution.

  ○ In order to have a solution, we need $f(x,y) = \sqrt{y-x}$ to be continuous in a rectangle containing (a,b). The function will not give real-number values if $x > y$, and it is not continuous near any point with $x = y$ either, because any rectangle around a point (a,a) will capture some points with $x > y$.

  ○ Thus, the solution is $\boxed{\text{guaranteed to exist only for } (a,b) \text{ with } b > a}$.

  ○ The partial derivative $f_y(x,y) = \dfrac{1}{2\sqrt{y-x}}$ is not defined if $x \geq y$ (since in addition to taking the square root of a negative number, we cannot divide by zero). So the solution is $\boxed{\text{unique for } (a,b) \text{ with } b > a}$.

  ○ Note: If we decide to allow complex numbers, then the function $f(x,y) = \sqrt{y-x}$ is defined and continuous on the entire plane (although it will take non-real values if $x > y$). So with this convention, the solution is $\boxed{\text{guaranteed to exist for all } (a,b)}$ because the function is now continuous everywhere. The derivative is discontinuous at points of the form (a,a) because dividing by zero is still not allowed, so the solution is guaranteed to be $\boxed{\text{unique for all } (a,b) \text{ with } a \neq b}$.

9.4   First-Order: Separable

- One type of first-order equations we can solve explicitly is the class of separable equations. Before giving the formal definition, we will give an example.

- Example: Solve the initial value problem $y' = 2xy$ with $y(1) = 1$.

  ○ We rearrange the equation as $\dfrac{y'}{y} = 2x$, and then integrate both sides.

  ○ This gives $\int \dfrac{y'}{y}\, dx = \int x\, dx = x^2 + C_1$.

  ○ In the left integral we can make the substitution $u = y(x)$, with $u' = y'\, dx$, to obtain $\ln(y) + C_2 = x^2 + C_1$.

  ○ Moving the constants around gives $\ln(y) = x^2 + C$ for some constant C.

  ○ Plugging in the condition $y(1) = 1$ gives $0 = 1^2 + C$, so $C = -1$.

  ○ Thus, $\boxed{\ln(y) = x^2 - 1}$, so that $y = \boxed{e^{x^2-1}}$.

  ○ Remark: We can simplify the procedure slightly if instead we convert the statement $\dfrac{dy}{dx} = 2xy$ into the statement $\dfrac{dy}{y} = 2x\, dx$. We can then integrate both sides directly, to obtain the statement $\ln(y) = x^2 + C$.

- Definition: A separable equation is of the form $y' = f(x) \cdot g(y)$ for some functions $f(x)$ and $g(y)$, or an equation equivalent to something of this form.

  ○ We can rearrange such an equation and then integrate both sides, in the same way as in the example above. We can simply the solving procedure slightly, as noted above: instead of making a substitution, we can use differentials.

- Here is the method for solving such equations:

○ <u>Step 1</u>: Replace y' with $\dfrac{dy}{dx}$, and then write the equation as $\dfrac{dy}{g(y)} = f(x)\,dx$.

○ <u>Step 2</u>: Integrate both sides (indefinitely), and place the $+C$ on the x side.

○ <u>Step 3</u>: If given, plug in the initial condition to solve for the constant C. (Otherwise, just leave it where it is.)

○ <u>Step 4</u>: Solve for y as a function of x, if possible.

- <u>Example</u>: Solve $y' = k \cdot y$, where k is a constant.

  ○ Step 1: Rewrite as $\dfrac{dy}{y} = k\,dx$.

  ○ Step 2: Integrate to get $\int \dfrac{dy}{y} = \int k\,dx$, which gives $\ln(y) = kx + C$.

  ○ Step 4: Exponentiate to get $y = \boxed{e^{kx+C} = C \cdot e^{kx}}$.

- <u>Example</u>: Solve the differential equation $y' = e^{x-y}$.

  ○ Step 1: Using the identity $e^{x-y} = e^x/e^y$, we can rewrite the equation as $e^y\,dy = e^x\,dx$.

  ○ Step 2: Integrate to get $\int e^y\,dy = \int e^x\,dx$, which gives $e^y = e^x + C$.

  ○ Step 4: Take the natural logarithm to get $y = \boxed{\ln(e^x + C)}$.

- <u>Example</u>: Find y given that $y' = x + xy^2$ and $y(0) = 1$.

  ○ Step 1: Rewrite as $\dfrac{dy}{1+y^2} = x\,dx$.

  ○ Step 2: Integrate to get $\int \dfrac{dy}{1+y^2} = \int x\,dx$, which gives $\tan^{-1}(y) = \dfrac{1}{2}x^2 + C$.

  ○ Step 3: Plug in the initial condition to get $\tan^{-1}(1) = C$, so that $C = \pi/4$.

  ○ Step 4: Taking the natural logarithm gives $y = \boxed{\tan\left(\dfrac{1}{2}x^2 + \dfrac{\pi}{4}\right)}$.

9.5   First-Order: Linear

- Another type of first-order equations we can solve explicitly is the class of first-order linear equations, which (upon dividing by the coefficient of y') can be written in the general form $y' + P(x) \cdot y = Q(x)$, where $P(x)$ and $Q(x)$ are some functions of x.

- It would be very convenient if we could just integrate both sides to solve the equation. However, in general, we cannot: the y' term is no difficulty, but the $P(x) \cdot y$ term causes trouble.

- To fix this issue, we use an "integrating factor": we multiply by a function $I(x)$ which will turn the left-hand side into the derivative of a single function.

  ○ What we would like to happen is for $I(x) \cdot y' + I(x)P(x) \cdot y$ to be the derivative of something nice.

  ○ When written this way, this sum looks sort of like the output of the product rule. If we can find $I(x)$ so that the derivative of $I(x)$ is $I(x)P(x)$, then this sum will be the derivative $\dfrac{d}{dx}[I(x) \cdot y]$.

  ○ What we want is $I(x)P(x) = I'(x)$. This is now a separable equation for the function $I(x)$, and we can see by inspection that one solution is $I(x) = e^{\int P(x)\,dx}$.

- Motivated by the above logic, here is the method for solving first-order linear equations:

  ○ <u>Step 1</u>: Put the equation into the form $y' + P(x) \cdot y = Q(x)$.

○ Step 2: Multiply both sides by the integrating factor $e^{\int P(x)\,dx}$ to get $e^{\int P(x)\,dx}y' + e^{\int P(x)\,dx}P(x)\cdot y = e^{\int P(x)\,dx}Q(x)$.

○ Step 3: Observe that the left-hand side is $\dfrac{d}{dx}\left[e^{\int P(x)\,dx}\cdot y\right]$, and take the antiderivative on both sides. (Don't forget the constant of integration C.)

○ Step 4: If given, plug in the initial condition to solve for the constant C. (Otherwise, just leave it where it is.)

○ Step 5: Solve for y as a function of x.

- Example: Find y given that $y' + 2xy = x$ and $y(0) = 1$.

  ○ Step 1: We have $P(x) = 2x$ and $Q(x) = x$.

  ○ Step 2: Multiply both sides by $e^{\int P(x)\,dx} = e^{x^2}$ to get $e^{x^2}y' + e^{x^2}\cdot 2x\cdot y = x\cdot e^{x^2}$.

  ○ Step 3: Taking the antiderivative on both sides yields $e^{x^2}y = \dfrac{1}{2}e^{x^2} + C$.

  ○ Step 4: Plugging in yields $e^0 \cdot 1 = \dfrac{1}{2}e^0 + C$ hence $C = \dfrac{1}{2}$.

  ○ Step 5: Solving for y gives $y = \boxed{\dfrac{1}{2} + \dfrac{1}{2}e^{-x^2}}$.

- Example: Find all functions y for which $xy' = x^4 - 4y$.

  ○ Step 1: We have $y' + \dfrac{4}{x}y = x^3$, so $P(x) = \dfrac{4}{x}$ and $Q(x) = x^3$.

  ○ Step 2: Multiply both sides by $e^{\int P(x)\,dx} = e^{4\ln(x)} = x^4$ to get $x^4 y' + 4x^3 y = x^7$,

  ○ Step 3: Taking the antiderivative on both sides yields $x^4 y = \dfrac{1}{8}x^8 + C$.

  ○ Step 5: Solving for y gives $y = \boxed{\dfrac{1}{8}x^4 + C\cdot x^{-4}}$.

- Example: Find y given that $y' \cdot \cot(x) = y + 2\cos(x)$ and $y(0) = -\dfrac{1}{2}$.

  ○ Step 1: We have $y' - y\tan(x) = 2\sin(x)$, with $P(x) = -\tan(x)$ and $Q(x) = 2\sin(x)$.

  ○ Step 2: Multiply both sides by $e^{\int P(x)\,dx} = e^{\ln(\cos(x))} = \cos(x)$ to get $y'\cdot\cos(x) - y\cdot\sin(x) = 2\sin(x)\cos(x)$.

  ○ Step 3: Taking the antiderivative on both sides yields $[y\cdot\cos(x)] = -\dfrac{1}{2}\cos(2x) + C$.

  ○ Step 4: Plugging in yields $-\dfrac{1}{2} = -\dfrac{1}{2}\cdot 1 + C$ hence $C = 0$.

  ○ Step 5: Solving for y gives $y = \boxed{-\dfrac{\cos(2x)}{2\cos(x)}}$.

9.6 First-Order: Autonomous Equations, Equilibria, and Stability

- An <u>autonomous equation</u> is a first-order equation of the form $\dfrac{dy}{dt} = f(y)$ for some function f.

  ○ An equation of this form is separable, and thus solvable in theory.

  ○ However, sometimes the function $f(y)$ is sufficiently complicated that we cannot actually solve the equation explicitly.

  ○ Nonetheless, would like to be able to say something about what the solutions look like, without actually solving the equation. Happily, this is possible.

- An <u>equilibrium solution</u>, also called a <u>steady state solution</u> or a <u>critical point</u>, is a solution of the form $y(t) = c$, for some constant c. (In other words, it is just a constant-valued solution.)

  ○ Clearly, if $y(t)$ is constant, then $y'(t)$ is zero everywhere. Thus, in order to find the equilibrium solutions to an autonomous equation $y' = f(y)$, we just need to solve $f(y) = 0$. (And this is not generally so hard.)

- For equilibrium solutions, we have some notions of "stability":

  ○ An equilibrium solution $y = c$ is <u>stable</u> <u>from above</u> if, when we solve $y' = f(y)$ with the initial condition $y(0) = c + \epsilon$ for some small but positive ϵ, the solution $y(t)$ moves toward c as t increases. This statement is equivalent to $f(c + \epsilon) < 0$.

  ○ A solution $y = c$ is <u>stable from below</u> if when we solve $y' = f(y)$ with the initial condition $y(0) = c - \epsilon$ for some small but positive ϵ, the solution $y(t)$ moves toward c as t increases. This statement is equivalent to $f(c - \epsilon) > 0$.

  ○ A solution $y = c$ is <u>unstable from above</u> if when we solve $y' = f(y)$ with the initial condition $y(0) = c + \epsilon$ for some small but positive ϵ, the solution $y(t)$ moves away from c as t increases. This statement is equivalent to $f(c + \epsilon) > 0$.

  ○ A solution $y = c$ is <u>unstable from below</u> if when we solve $y' = f(y)$ with the initial condition $y(0) = c - \epsilon$ for some small but positive ϵ, the solution $y(t)$ moves away from c as t increases. This statement is equivalent to $f(c - \epsilon) < 0$.

- We say a solution is <u>stable</u> if it is stable from above and from below. We say it is <u>unstable</u> if it unstable from above and from below. Otherwise (if it is stable from one side and unstable from the other) we say it is <u>semistable</u>.

- From the equivalent conditions about the sign of f, here are the steps to follow to find and classify the equilibrium states of $y' = f(y)$:

  ○ <u>Step 1</u>: Find all values of c for which $f(c) = 0$, to find the equilibrium states.

  ○ <u>Step 2</u>: Mark all the equilibrium values on a number line, and then in each interval between two critical points, plug in a test value to f to determine whether f is positive or negative on that interval.

  ○ <u>Step 3</u>: On each interval where f is positive, draw right-arrows, and on each interval where f is negative, draw left-arrows.

  ○ <u>Step 4</u>: Using the arrows, classify each critical point: if the arrows point toward it from both sides, it is stable. If the arrows point away, it is unstable. If the arrows both point left or both point right, it is semistable.

  ○ <u>Step 5</u> (optional): Draw some solution curves, either by solving the equation or by using the stability information.

- <u>Example</u>: Find the equilibrium states of $y' = y$ and determine stability.

  ○ Step 1: We have $f(y) = y$, which obviously is zero only when $y = 0$.

  ○ Step 2: We draw the line and plug in 2 test points to see that the sign diagram looks like $\ominus | \oplus$.
  $$_0$$

  ○ Step 3: Changing the diagram to arrows gives $\leftarrow | \rightarrow$.
  $$_0$$

  ○ Step 4: So we can see from the diagram that the only equilibrium point $\boxed{0 \text{ is unstable}}$.

  ○ Step 5: We can of course solve the equation to see that the solutions are of the form $y(t) = C\, e^t$, and indeed, the equilibrium solution $y = 0$ is unstable:
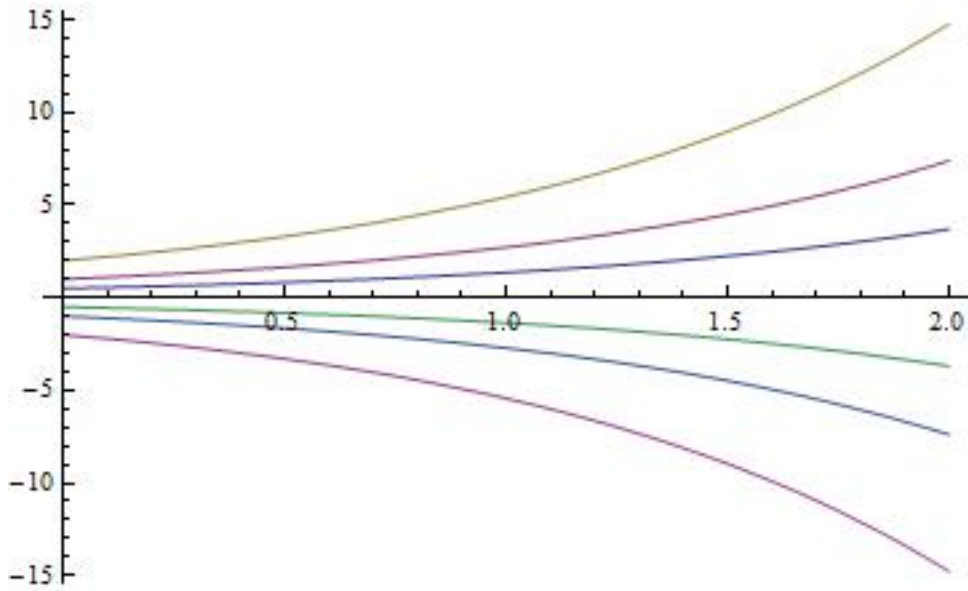
- <u>Example</u>: Find the equilibrium states of $y' = y^2(y-1)(y-2)$ and determine stability.

  - Step 1: We have $f(y) = y^2(y-1)(y-2)$, which conveniently is factored. We see it is zero when $y = 0$, $y = 1$, and $y = 2$.

  - Step 2: We draw the line and plug in 4 test points to see that the sign diagram looks like $\oplus| \underset{0}{\ } \oplus | \underset{1}{\ } \ominus |\underset{2}{\ } \oplus.$

  - Step 3: Changing the diagram to arrows gives $\rightarrow | \underset{0}{\ } \rightarrow | \underset{1}{\ } \leftarrow | \underset{2}{\ } \rightarrow.$

  - Step 4: So we can see from the diagram that $\boxed{0 \text{ is semistable}}$, $\boxed{1 \text{ is stable}}$, and $\boxed{2 \text{ is unstable}}$.

  - Step 5: In this case, it is possible to obtain an implicit solution by integration; however, an explicit solution does not exist. However, we can graph some solution curves to see, indeed, our classification is accurate:
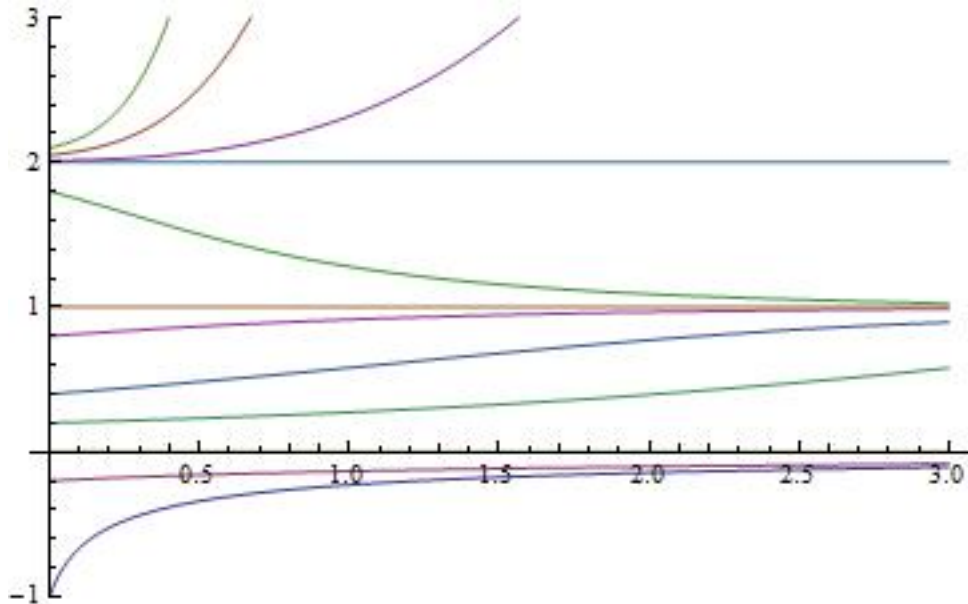
9.7   First Order: Some Applications

9.7.1   The Logistic Equation

- <u>Example</u>: Solve the differential equation $P' = kP(M - P)$, where k and M are positive constants.

○ Step 1: Rewrite as $\dfrac{M\,dP}{P(M-P)} = kM\,dt$.

○ Step 2: Integrate both sides to obtain $\int \dfrac{M}{P(M-P)}\,dP = \int kM\,dt$. To evaluate the P-integral, use partial fraction decomposition: $\dfrac{M}{P(M-P)} = \dfrac{1}{P} + \dfrac{1}{M-P}$. Evaluating the integrals therefore yields $\ln(P) - \ln(M-P) = kMt + C$.

○ Step 4: Combine the logarithms to obtain $\ln\left(\dfrac{P}{M-P}\right) = kMt + C$; now exponentiate to get $\dfrac{P}{M-P} = Ce^{kMt}$. Solving for P yields, finally, $\boxed{P(t) = \dfrac{M}{1 + Ce^{-kMt}}}$.

○ <u>Note</u>: If we want to satisfy the initial condition $P(0) = P_0$, then plugging in shows $C = \dfrac{M}{P_0} - 1$. Then the solution can be rewritten in the form $\boxed{P(t) = \dfrac{MP_0}{P_0 + (M-P_0)e^{-kMt}}}$.

• <u>Remark</u>: Differential equations of this form are called logistic equations. With the explicit solution given here, or using the properties of stable and unstable equilibria, we can observe some properties of the solution curves.

   ○ We can see that $P = 0$ and $P = M$ are the only equilibrium solutions.

   ○ Since k and M are positive, the sign diagram is $\ominus|\ \oplus\ |\ \ominus$. So 0 is an unstable equilibrium and M is a stable equilibrium.

   ○ Thus, as $t \to \infty$, as long as the starting population P_0 is positive, the population $P(t)$ tends toward the "carrying capacity" of M. (We can also see this using the explicit solution.)

• <u>Remark</u>: If instead we had $k < 0$ rather than $k > 0$, the stability of the two equilibrium solutions would flip (0 would be stable and M would be unstable). Then we would be in an "extinction-explosion" scenario: if the initial population P_0 were less than M, it would tend toward 0, and if the population were greater than M, it would tend toward ∞.

9.7.2   Mixing Problems

• The setup of the general mixing problem is as follows:

   ○ We have some reservoir (pool, lake, ocean, planet, room) of liquid (water, gas) which has some substance (pollution, solute) dissolved in it.

   ○ The reservoir starts at an initial volume V_0 and there is an initial amount of substance y_0 in the reservoir.

   ○ We have some amount of liquid $\mathrm{In}(t)$ flowing in with a given concentration $k(t)$ of the substance, and some other amount of liquid $\mathrm{Out}(t)$ flowing out.

   ○ We assume that the substance is uniformly and perfectly mixed in the reservoir, and want to know the amount $y(t)$ of the substance that remains in the reservoir after time t.

• Note that this is the <u>general</u> setup. In more specific examples, the amount of liquid flowing in or out may be constants (i.e., not depending on time), and similarly the concentration of the liquid flowing in could also be a constant. The solution is the same, of course.

• We can solve the problem as follows:

   ○ Let $V(t)$ be the total volume of the reservoir. If $y(t)$ is the total amount of substance in the reservoir, the concentration of substance in the reservoir is $\dfrac{y(t)}{V(t)}$. Thus the total amount of substance moving in is $k \cdot \mathrm{In}(t)$ and the total amount of substance moving out is the concentration of substance times the volume moving out, or $\dfrac{y(t)}{V(t)} \cdot \mathrm{Out}(t)$.

○ We have $V'(t) = \text{In}(t) - \text{Out}(t)$, and $y'(t) = k(t) \cdot \text{In}(t) - \dfrac{y(t)}{V(t)} \cdot \text{Out}(t)$. For clarity, refer to the diagram:
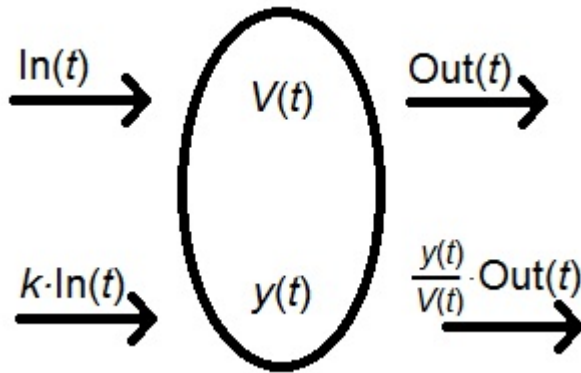
○ To solve this system, we can integrate to find $V(t)$ explicitly.

○ Then we can rewrite the other equation as $y' + \dfrac{\text{Out}(t)}{V(t)} \cdot y = k(t) \cdot \text{In}(t)$, which we can solve because it is first-order linear.

- <u>Example</u>: A mixing tank initially contains 10L of a salt water solution with a salt concentration of 5g/L. Pure water flows into the tank at a rate of 3L/s, and the mixed solution flows out at a rate of 2L/s. Find the concentration of salt as a function of time.

  ○ We have $V(0) = 10\text{L}$ and $y(0) = 10\text{L} \cdot 5\text{g/L} = 50\text{g}$, and $k(t) = 0$.

  ○ We also have $\text{In}(t) = 3\text{L/s}$ and $\text{Out}(t) = 2\text{L/s}$, so $V'(t) = 3\text{L/s} - 2\text{L/s} = 1\text{L/s}$.

  ○ Integrating gives $V(t) = (C + t)\text{L}$. Since $V(0) = 10\text{L}$, we get $V(t) = (10 + t)\text{L}$.

  ○ Furthermore, $y'(t) = 0 \cdot 3\text{L/s} - \dfrac{y(t)}{(10 + t)\text{L}} \cdot 2\text{L/s}$, which is equivalent to $y' = -\dfrac{y}{10 + t} \cdot 2$.

  ○ This is a separable equation. We get $\dfrac{dy}{dt} = \dfrac{-y}{10 + t} \cdot 2$, so that $\dfrac{dy}{y} = -\dfrac{2}{10 + t} \, dt$.

  ○ Integrating gives $\displaystyle\int \dfrac{dy}{y} = \int -\dfrac{2}{10 + t} \, dt$, so $\ln(y) = -2\ln(10 + t) + C$.

  ○ Since $y(0) = 50\text{g}$, we see that $\ln(50) = -2\ln(10) + C$, so $C = \ln(50) + 2\ln(10) = \ln(5000)$.

  ○ Then $\ln(y) = -2\ln(10 + t) + \ln(5000) = \ln\left[5000(10 + t)^{-2}\right]$, which is equivalent to $y = 5000(10 + t)^{-2}\text{g}$. The concentration is equal to $\dfrac{y(t)}{V(t)} = \boxed{5000(10 + t)^{-3}\text{g/L}}$.

Well, you're at the end of my handout. Hope it was helpful.