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8 Power Series and Taylor Series

In this chapter, we continue our discussion of infinite series from the previous chapter. However, we will narrow our focus to a particular kind of infinite series, called a power series, which has the general form $\sum_{n=0}^{\infty} a_n(x - c)^n$ for real numbers a_n and c , and a parameter x . We will discuss the basic theory of power series and methods for representing functions as power series.

We will then turn our attention to Taylor series, which are a special type of power series that arise in trying to find good polynomial approximations to arbitrary functions, and conclude by outlining some of the more important applications of Taylor series.

8.1 Power Series

- **Definition:** A power series centered at $x = c$ is a series of the form $\sum_{n=0}^{\infty} a_n(x - c)^n$, where the center c and the coefficients a_i are constants.

- We will usually be interested in power series centered at $x = 0$, which have the simpler form $\sum_{n=0}^{\infty} a_n x^n$.

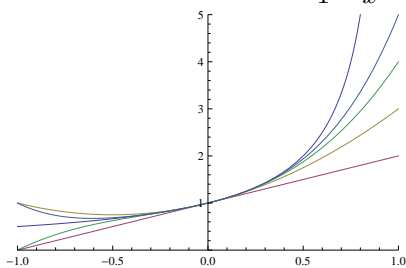
- We will generally think of a given power series $\sum_{n=0}^{\infty} a_n(x - c)^n$ as a function of x .

- Our initial goal is to study for which x this power series converges.

- We will then turn our attention to describing the resulting function of x (defined where the series converges).

- Example: The geometric series $\sum_{n=0}^{\infty} x^n$ is a power series centered at $x = 0$ all of whose coefficients are equal to 1.

- From our earlier analysis of geometric series, we know this series will converge to the limit $\frac{1}{1-x}$ whenever $-1 < x < 1$, and that it will diverge for other x .
- By the definition of convergent series, this says that the sequence of polynomials $1, 1+x, 1+x+x^2, 1+x+x^2+x^3, \dots$ converges to the value $\frac{1}{1-x}$ whenever $-1 < x < 1$.
- We can see this convergence explicitly from the graphs (the functions are $1+x, 1+x+x^2, 1+x+x^2+x^3, 1+x+x^2+x^3+x^4$, and $\frac{1}{1-x}$ from bottom to top):



8.1.1 Convergence of Power Series

- In general, we can typically determine where a power series converges by using the Ratio or Root Tests. A useful technique is to combine the Ratio/Root Tests with the Absolute Convergence Theorem to obtain versions which apply to general series (possibly with negative terms):

- Strengthened Ratio Test: If $\{b_n\}$ has the property that $\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right|$ exists and equals some constant ρ , then the sum $\sum_{n=1}^{\infty} b_n$ converges if $\rho < 1$, and diverges if $\rho > 1$.
- Strengthened Root Test: If $\{b_n\}$ has the property that $\lim_{n \rightarrow \infty} \sqrt[n]{|b_n|}$ exists and equals some constant ρ , then the sum $\sum_{n=1}^{\infty} b_n$ converges if $\rho < 1$, and diverges if $\rho > 1$.
- For both tests, if $\rho = 1$ then the test is inconclusive, while if $\rho = \infty$ then the series diverges.

- Example: Determine the values of x for which the power series $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ converges.

- We use the Ratio Test: with $b_n = \frac{x^n}{n^2}$, we have $\left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{x^{n+1}/(n+1)^2}{x^n/n^2} \right| = |x| \cdot \left(\frac{n+1}{n} \right)^2$.
- Then $\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = |x|$.
- Thus, by the (strengthened) Ratio Test, we see that the series converges whenever $|x| < 1$ and diverges whenever $|x| > 1$.
- There are two cases where the test is inconclusive: $x = 1$ and $x = -1$.
- When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p -series.

- When $x = -1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which converges by the Alternating Series Test (or by the fact that its absolute value series is the one we just saw above).
 - Therefore, the power series converges for $-1 \leq x \leq 1$ and diverges for other x .
- Example:** Determine the values of x for which the power series $\sum_{n=1}^{\infty} \frac{1}{3^n} (x-2)^n$ converges.
 - We use the Root Test: with $b_n = \frac{(x-2)^n}{3^n}$, we have $\sqrt[n]{|b_n|} = \left| \frac{x-2}{3} \right|$.
 - Then $\lim_{n \rightarrow \infty} \sqrt[n]{|b_n|} = \left| \frac{x-2}{3} \right|$.
 - Thus, by the (strengthened) Ratio Test, we see that the series converges whenever $\left| \frac{x-2}{3} \right| < 1$ and diverges whenever $\left| \frac{x-2}{3} \right| > 1$, while the test is inconclusive when $\left| \frac{x-2}{3} \right| = 1$.
 - Notice that $\left| \frac{x-2}{3} \right| < 1$ is equivalent to $-1 < \frac{x-2}{3} < 1$, which is the same as $-1 < x < 5$.
 - When $x = -1$, the series is $\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n$, which diverges.
 - When $x = 5$, the series is $\sum_{n=1}^{\infty} \frac{3^n}{3^n} = \sum_{n=1}^{\infty} 1$, which also diverges.
 - Therefore, the power series converges for $-1 < x < 5$ and diverges for other x .
 - Notice, in particular, that the region of convergence is the interval $(-1, 5)$, and its midpoint is the center of the power series.
- Example:** Determine the values of x for which the power series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges.
 - We use the Ratio Test: with $b_n = \frac{1}{n!} x^n$, we have $\left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \left| x \cdot \frac{n!}{(n+1)!} \right| = \frac{|x|}{n+1}$.
 - Then, for any fixed value of x , we see that $\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0$.
 - Thus, by the (strengthened) Ratio Test, we see that the series converges for all x .
- Example:** Determine the values of x for which the power series $\sum_{n=0}^{\infty} n! \cdot x^n$ converges.
 - We use the Ratio Test: with $b_n = n! \cdot x^n$, we have $\left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{(n+1)! \cdot x^{n+1}}{n! \cdot x^n} \right| = \left| x \cdot \frac{(n+1)!}{n!} \right| = |x| \cdot (n+1)$.
 - Then, for any nonzero value of x , we see that $\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} (n+1)|x| = +\infty$. If $x = 0$, then the limit is clearly zero.
 - Thus, by the (strengthened) Ratio Test, we see that the series converges only for $x = 0$.
- In each of the examples above, notice that the set of x for which the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converged was an interval whose midpoint was the center $x = c$ of the power series.

- Note that we have included the case of the degenerate “interval” $[0, 0]$ consisting of a single point, as well as the infinite interval $(-\infty, \infty)$.
- In fact, the region of convergence is always an interval. To show this, we first need a preliminary result:
- **Proposition:** If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = d$, then it converges absolutely for all x with $|x| < |d|$.

- **Proof:** Since the series $\sum_{n=0}^{\infty} a_n d^n$ converges, by the “Test for Divergence” we know that $\lim_{n \rightarrow \infty} a_n d^n = 0$.
- By the definition of limit, in particular this implies that for large enough N , $|a_N d^N| \leq 1$: therefore, $|a_N| \leq d^{-N}$.
- But then $\sum_{n=N}^{\infty} |a_n x^n| \leq \sum_{n=N}^{\infty} \left| \frac{x}{d} \right|^n$, and this last series is a convergent geometric series when $\left| \frac{x}{d} \right| < 1$. This implies the original series converges absolutely for $\left| \frac{x}{d} \right| < 1$ – i.e., whenever $|x| < d$.

- From this, we can conclude that the set of convergence of a power series must have a very particular form:
- **Theorem (Power Series Convergence):** For any power series $\sum_{n=0}^{\infty} a_n (x - c)^n$, precisely one of the following three things holds:

1. The series converges absolutely for every x .
2. The series converges at $x = c$ and diverges for other x .
3. There exists a positive real number R such that the series converges absolutely for x with $|x - c| < R$ and diverges for $|x - c| > R$. The series may or may not converge at the two endpoints $x = c \pm R$.

- **Remark:** The value of R is called the radius of convergence for the power series. It is conventional to say that $R = \infty$ in the first case and to say that $R = 0$ in the second case. (Thus, for example, the radius of convergence of the power series $\sum_{n=0}^{\infty} x^n$ is 1, while the radius of convergence of $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ is ∞ .)

- **Proof:** Let $u = x - c$: then the power series has the form $\sum_{n=0}^{\infty} a_n u^n$.

- * Consider the set of values of $|u|$ such that this series converges: if there is no upper bound, then (by the previous proposition applied to an increasing sequence of values of u) we conclude that the series converges absolutely for every value of u .
- * Otherwise, there is some upper bound on the values of u where the series converges, hence (by an axiomatic property of the real numbers) there is some least upper bound R . If $R = 0$, then the series converges only when $u = 0$ – namely, for $x = c$.
- * If $R > 0$, then the series converges for values of u an arbitrarily small distance below R . By applying the proposition above, we conclude that the series converges absolutely whenever $|u| < R$ – namely, for $|x - c| < R$.
- * Finally, by the definition of R , the series will diverge whenever $|u| > R$. (Otherwise, R would not be the least upper bound of the set of values of u where the series converges.)

- **Example:** Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{2n+1}$, and also determine where it converges absolutely, where it converges conditionally, and where it diverges.

- We use the Ratio Test: with $b_n = \frac{x^n}{2n+1}$, we have $\left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{x^{n+1}/(2n+3)}{x^n/(2n+1)} \right| = |x| \cdot \left(\frac{2n+1}{2n+3} \right)$.

- Then $\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = |x|$.
- Thus, by the (strengthened) Ratio Test, we see that the series converges whenever $|x| < 1$ and diverges whenever $|x| > 1$.
- By the Power Series Convergence Theorem, we conclude that the radius of convergence is 1 and that the series is absolutely convergent when $|x| < 1$.
- There are two cases where the test is inconclusive: $x = 1$ and $x = -1$.
- When $x = 1$, the series is $\sum_{n=0}^{\infty} \frac{1}{2n+1}$. This series diverges, either by the Integral Test, or by comparison to the divergent p -series $\sum_{n=1}^{\infty} \frac{1}{n}$.
- When $x = -1$, the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. It is easy to check that this series converges by the Alternating Series Test. Its absolute value series is the divergent series we just analyzed above, so the series is conditionally convergent.
- Therefore, $\sum_{n=0}^{\infty} \frac{x^n}{2n+1}$ is absolutely convergent for $-1 < x < 1$, conditionally convergent at $x = -1$, and divergent for $x < -1$ and $x \geq 1$.

8.1.2 Power Series as Functions

- We now consider the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ as a function $f(x)$, defined inside its interval of convergence. Our first goal is to determine whether $f(x)$ is differentiable, and how to compute its derivatives:
- **Theorem** (Power Series Differentiation): If the power series $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ has radius of convergence $R > 0$, then the function $f(x)$ is differentiable and its derivative is $f'(x) = \sum_{n=1}^{\infty} n a_n(x-c)^{n-1}$, valid for $|x-c| < R$. The radius of convergence of the power series for $f'(x)$ is also equal to R .
 - What this means is that we can compute the derivative of a power series simply by differentiating “term-by-term”: namely, by taking the derivative of each term of the power series individually, and then summing them.
 - Observe that the expression for the derivative can equivalently be written as $f'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1}(x-c)^n$.
 - * This result is obtained simply by shifting the index n by 1. One reason this formulation is sometimes preferable is because the power of $(x-c)$ is n rather than $n-1$.
 - The proof of this theorem follows by manipulating the difference quotient for the derivative and rearranging the terms of the relevant infinite series, which is allowable since they converge absolutely. (We will omit the precise technical details, since they are not especially enlightening.)
- Since the radius of convergence for $f'(x)$ is the same as for $f(x)$, iterating the result of the Theorem shows that f in fact has derivatives of *all* orders, and that they all have radius of convergence R around $x = c$.
- **Example**: Find the derivative of the power series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$.

- We simply differentiate term-by-term to obtain the series $\sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \dots$.

- We can in fact sum this series by breaking it apart as a sum of geometric series, each of which we can compute:

$$\begin{aligned}
 1 + 2x + 3x^2 + 4x^3 + \dots &= (1 + x + x^2 + x^3 + \dots) + (x + x^2 + x^3 + \dots) + (x^2 + x^3 + \dots) + (x^3 + \dots) + \dots \\
 &= \frac{1}{1-x} + \frac{x}{1-x} + \frac{x^2}{1-x} + \frac{x^3}{1-x} + \dots \\
 &= \frac{1 + x + x^2 + x^3 + \dots}{1-x} = \frac{1/(1-x)}{1-x} = \frac{1}{(1-x)^2}
 \end{aligned}$$

and these manipulations are valid whenever $|x| < 1$, since all we needed was for the series $1 + x + x^2 + x^3 + \dots$ to converge.

- Indeed, since we know that the geometric series $\sum_{n=0}^{\infty} x^n$ converges to the function $f(x) = \frac{1}{1-x}$ when $|x| < 1$, the theorem above tells us that the series for the derivative should converge to the actual derivative $f'(x) = \frac{1}{(1-x)^2}$ whenever $|x| < 1$. (And this is precisely what we obtained above!)

- Notice, in particular, that we can evaluate some new infinite series by plugging in specific values of x .

For example, setting $x = \frac{1}{2}$ yields $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = f'(\frac{1}{2}) = \frac{1}{(1-1/2)^2} = 4$.

- In other words, the infinite series $1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \dots$ has sum equal to 4.

- **Example:** Find the derivative of the power series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$.

- With $a_n = \frac{1}{n!}$, we apply the “shifted” form to see that

$$\begin{aligned}
 f'(x) &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} \frac{n+1}{(n+1)!} x^n \\
 &= \boxed{\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots}
 \end{aligned}$$

- Notice in particular that the derivative series is exactly the same as the original series! In other words, this function satisfies the differential equation $f'(x) = f(x)$. (As we will see soon, this is because $f(x)$ is actually equal to e^x .)

- Now that we have analyzed how to differentiate power series, we can ask about integrating power series. This turns out to be just as straightforward:

- **Theorem (Power Series Integration):** If the power series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has radius of convergence

$R > 0$, then the function $f(x)$ has general antiderivative $\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1}$, valid for $|x-c| < R$. The radius of convergence of the power series for the antiderivative is also equal to R .

- In other words, we can integrate a power series simply “term-by-term”, in the same way we can take a derivative.

- By shifting the index n by 1, the expression for the antiderivative can also be written as $\int f(x) dx =$

$$C + \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} (x-c)^n.$$

- The proof of this theorem mostly follows from the differentiation theorem: all that needs to be verified is that the power series expression for the antiderivative actually converges for $|x-c| < R$, since differentiating it term-by-term clearly gives $f(x)$. (Again, we will omit the technical details.)

- Example: Find a power series expansion for $\tan^{-1}(x)$ centered at $x = 0$.

- Recalling the fact that $\tan^{-1}(x)$ is an antiderivative of $\frac{1}{1+x^2}$, we will first find a power series expansion for $\frac{1}{1+x^2}$, and then integrate it.

- From the geometric series expansion $\frac{1}{1-y} = 1 + y + y^2 + y^3 + \dots = \sum_{n=0}^{\infty} y^n$, we set $y = -x^2$ to obtain $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, valid for $|x| < 1$.

- Now we integrate both sides to obtain $\tan^{-1}(x) = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$, for some constant C .

- If we set $x = 0$, we see that $C = 0$, so our result is $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$, valid for $|x| < 1$.

- Another question we could ask is: can we do arithmetic with power series? It turns out the answer is yes:

- Theorem (Power Series Arithmetic): If the power series $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ and $\sum_{n=0}^{\infty} b_n(x-c)^n$ both converge absolutely for $|x-c| < R$, then the function $f(x) + g(x)$ has power series $\sum_{n=0}^{\infty} [a_n + b_n](x-c)^n$, and

the function $f(x) \cdot g(x)$ has power series $\sum_{n=0}^{\infty} d_n(x-c)^n$, where $d_n = a_0b_n + a_1b_{n-1} + \dots + a_{n-1}b_1 + a_nb_0 = \sum_{k=0}^n a_k b_{n-k}$, both convergent absolutely for $|x-c| < R$.

- The first part of the theorem says that we can add two power series term by term.
- The second part of the theorem says that we can multiply two power series essentially using the distributive law, as long as we collect the terms appropriately. (The expression for the coefficients are exactly the same as those given by multiplying out two polynomials.)
- Both results essentially follow from the fact that we may rearrange terms in an absolutely convergent series. (We will again omit the details, since they are not enlightening.)

- Example: Find a power series expansion for $x^2 \tan^{-1}(x)$ centered at $x = 0$.

- We use the multiplication result on $f(x) = x^2$ and $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$.

- In this case, we immediately see that the power series is simply the series for $\tan^{-1}(x)$ but with each term multiplied by x^2 : $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+3} = x^3 - \frac{x^5}{3} + \frac{x^7}{5} - \frac{x^9}{7} + \dots$.

- Example: Find a power series expansion for $\frac{1}{(1-x)^2}$ centered at $x = 0$.

- We use the multiplication result on $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$ and $g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

- Here, $a_n = b_n = 1$ for all n , so we easily can find that $d_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0 = \underbrace{1 + 1 + \dots + 1}_{n+1 \text{ ones}} = n + 1$.

- Therefore, we conclude that the power series for $\frac{1}{(1-x)^2}$ is given by $\sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \dots$.

- Observe that we already found this same result earlier, by differentiating the series $\frac{1}{1-x}$ term-by-term.

- Example:** Find the terms through degree 4 in a power series expansion for $\frac{\tan^{-1}(x)}{(1-x)^2}$ centered at $x = 0$.

- We use the multiplication result on $f(x) = \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ and

$$g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \dots$$

- We could in principle write down a general expression for the terms in the product series using the formula.
- However, with complicated series, it is often faster just to multiply out the product of the two power series using the distributive law, and then collect terms of each degree (which ultimately amounts to the same thing, but is easier to do by hand).
- Explicitly multiplying out $\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) \cdot (1 + 2x + 3x^2 + 4x^3 + \dots)$ yields

$$\begin{aligned} & (x + 2x^2 + 3x^3 + 4x^4 + \dots) + \left(-\frac{x^3}{3} - \frac{2x^4}{3} - \frac{3x^5}{3} - \frac{4x^6}{3}\right) \\ & + \left(\frac{x^6}{5} + \frac{2x^7}{5} + \frac{3x^8}{5} + \frac{4x^9}{5} + \dots\right) + \left(-\frac{x^8}{7} - \frac{2x^9}{7} - \frac{3x^{10}}{7} - \frac{4x^{11}}{7} + \dots\right) + \dots \end{aligned}$$

and collecting terms of the same degree gives an expansion starting $x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \dots$.

- Thus, the power series expansion for $\frac{\tan^{-1}(x)}{(1-x)^2}$ at $x = 0$ begins $x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \dots$.

8.2 Taylor Series

- We have already seen that a few functions can be represented as power series, such as the function $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, and that any power series is infinitely differentiable on its interval of convergence. We would now like to turn the question the other way around, namely: if we have an infinitely differentiable function $f(x)$, can it be represented as a power series, and if so, what are its coefficients?

- Suppose that $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \dots$ is representable by a power series with a positive radius of convergence.

- Then, by differentiating term-by-term, we obtain

$$\begin{aligned} f(x) &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \dots \\ f'(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \dots \\ f''(x) &= 2a_2 + 3 \cdot 2 \cdot a_3(x-c) + 4 \cdot 3 \cdot a_4(x-c)^2 + 5 \cdot 4 \cdot a_5(x-c)^3 + \dots \\ f'''(x) &= 3 \cdot 2 \cdot a_3 + 4 \cdot 3 \cdot 2 \cdot a_4(x-c) + 5 \cdot 4 \cdot 3 \cdot a_5(x-c)^2 + \dots \\ f''''(x) &= 4 \cdot 3 \cdot 2 \cdot a_4 + 5 \cdot 4 \cdot 3 \cdot 2 \cdot a_5(x-c) + \dots \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

- Now, if we set $x = c$ in each of these expressions (which is allowed because $x = c$ is always inside the interval of convergence), all of the positive-degree terms will vanish, and we obtain the equalities

$$\begin{aligned} f(c) &= a_0 \\ f'(c) &= a_1 \\ f''(c) &= 2a_2 \\ f'''(c) &= 3 \cdot 2 \cdot a_3 \\ f^{(4)}(c) &= 4 \cdot 3 \cdot 2 \cdot a_4 \\ &\vdots \quad \vdots \quad \vdots \end{aligned}$$

- We see immediately that a_n must necessarily be equal to $\frac{f^{(n)}(c)}{n!}$, where $f^{(n)}$ denotes the n th derivative (as always).
- Thus, if the function $f(x)$ can be represented as a power series (which is not necessarily the case!), the series must be of the form $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n$.
- It therefore seems worthwhile to study power series having this form.
- Definition:** If f is a function whose n th derivative at $x = a$ exists for every n , then the Taylor series for $f(x)$ at $x = a$ is the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$, where $f^{(n)}$ is the n th derivative of f .

- The first few terms of the expansion are $f(a) + f'(a) \cdot (x - a) + \frac{f''(a)}{2} (x - a)^2 + \frac{f'''(a)}{6} (x - a)^3 + \dots$.
- It is most common to deal with Taylor series at $x = 0$, since the terms in the power series are simplest to write in this case. Such series are often called Maclaurin series (though there is no need to give them a special name).
- Since Taylor series are special cases of power series, all of our results about power series automatically apply to them: thus, for example, the Taylor series for $f'(x)$ is obtained by differentiating the Taylor series for $f(x)$ term-by-term.
- At the moment, we do not have the ability to determine whether the Taylor series for a function actually converges to that function or not. (We will return to this question soon.)

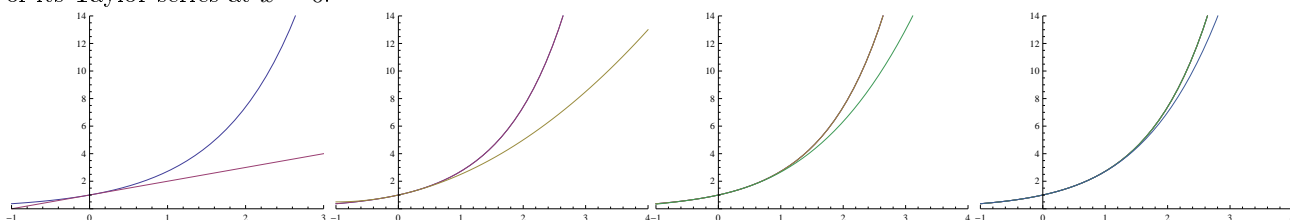
- Example:** Find the Taylor series for $f(x) = e^x$ at $x = 0$, and at $x = 1$.

- Since $f^{(n)}(x) = e^x$ for all n , we have $f^{(n)}(0) = 1$ and $f^{(n)}(1) = e$ for all n .

- Thus, the Taylor series for e^x at $x = 0$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$, while the Taylor series

for e^x at $x = 1$ is $\sum_{n=0}^{\infty} \frac{e}{n!} (x - 1)^n = e + e(x - 1) + \frac{e^2}{2!} (x - 1)^2 + \dots$.

- Here are plots of $y = e^x$ along with the partial sums $1 + x$, $1 + x + \frac{x^2}{2!}$, $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, and $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$ of its Taylor series at $x = 0$:



- As can be seen from the graphs, the successive partial sums approach the graph more and more closely.

- **Example:** Find the Taylor series for $f(x) = \sin(x)$ and $g(x) = \cos(x)$ at $x = 0$.

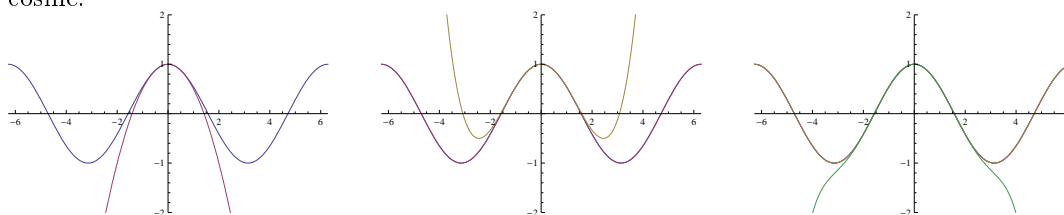
- For sine, we have $f(x) = \sin(x)$, $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$, $f^{(4)}(x) = \sin(x)$, $f^{(5)}(x) = \cos(x)$, and so forth. Evaluating at $x = 0$ yields (respectively) the values 0, 1, 0, -1, 0, 1, 0, -1, ...

- Thus, the first few terms in the Taylor series for $\sin(x)$ are $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$.

- To write the general term of the series, we note that only odd-degree terms show up, and they alternate in sign. Writing down the terms yields the expression
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

- For cosine, we can simply take the derivative term-by-term to obtain
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

- Here are plots of the partial sums $1 - \frac{x^2}{2!}$, $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, and $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$ for the Taylor series for cosine:



- As can be seen from the graphs, the successive Taylor series align with the graph on larger and larger intervals around 0.

- **Example:** Find the Taylor series for $f(x) = (1+x)^k$ at $x = 0$, in terms of k .

- We have $f'(x) = k \cdot (1+x)^{k-1}$, $f''(x) = k(k-1) \cdot (1+x)^{k-2}$, $f'''(x) = k(k-1)(k-2) \cdot (1+x)^{k-3}$, and in general, $f^{(n)}(x) = k(k-1) \cdots (k-n+1) \cdot (1+x)^{k-n}$.

- Thus, $f^{(n)}(0) = k(k-1) \cdots (k-n+1)$, so the n th Taylor coefficient is $a_n = \frac{k(k-1) \cdots (k-n+1)}{n!}$.

- The traditional notation for the expression $a_n = \frac{k(k-1) \cdots (k-n+1)}{n!}$ is $\binom{k}{n}$, and it is called a binomial coefficient.

- Thus, the Taylor series is
$$\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots$$
 and it is known as the binomial series.

- For k a positive integer, the coefficients are eventually zero (starting with the $k+1$ st coefficient, since it contains a $k-k$ term in the numerator), and we will recover the finite binomial expansion. For example, writing out the series with $k = 2$ yields the familiar $(1+x)^2 = 1 + 2x + x^2$.

- **Example:** Find the Taylor series for $f(x) = \frac{1}{x}$ at $x = 3$.

- We have $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f'''(x) = \frac{-3!}{x^4}$, $f^{(4)}(x) = \frac{4!}{x^5}$, $f^{(5)}(x) = \frac{-5!}{x^6}$, and so forth.

- Following the pattern indicates that the n th derivative is $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$, meaning that $f^{(n)}(3) = \frac{(-1)^n n!}{3^{n+1}}$.

- Thus, the Taylor series for $f(x) = \frac{1}{x}$ at $x = 3$ is
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-3)^n = 1 - \frac{1}{3}(x-3) + \frac{1}{9}(x-3)^2 - \frac{1}{27}(x-3)^3 + \dots$$

- **Example:** Find the Taylor series for $f(x) = \frac{1}{1-x}$ at $x = 0$.
 - We have $f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2}{(1-x)^3}$, $f'''(x) = \frac{3!}{(1-x)^4}$, and in general, $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$, meaning that $f^{(n)}(0) = n!$.
 - Thus, the Taylor series is simply $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$.
 - As we have already seen several times, this power series does actually converge to the value $\frac{1}{1-x}$ for $|x| < 1$.

- **Example:** Find the Taylor series for $f(x) = x^3 \sin(x)$ at $x = 0$, and then find $f^{(10)}(0)$.

- This is simply x^3 times the series for $\sin(x)$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+4} = x^4 - \frac{x^6}{3!} + \frac{x^8}{5!} - \frac{x^{10}}{7!} + \dots$.

- From the series expansion, we know immediately that the value of the tenth derivative $f^{(10)}(0)$ at $x = 0$ is equal to $10!$ times the coefficient of x^{10} : thus, $f^{(10)}(0) = -\frac{10!}{7!} = \boxed{-720}$.

- Note that it is not at all easy to compute the value of $f^{(10)}(0)$ simply by taking derivatives: we have $f'(x) = x^3 \cos(x) + 3x^2 \sin(x)$, $f''(x) = -x^3 \sin(x) + 6x^2 \cos(x) + 6x \sin(x)$, $f'''(x) = -x^3 \cos(x) - 9x^2 \sin(x) + 18x \cos(x) + 6 \sin(x)$, and each subsequent derivative requires three more applications of the Product Rule.

- **Example:** Find the Taylor series for $e^x \sin(x)$ at $x = 0$ up through degree 5.

- We can obtain this series by multiplying out the series for e^x with the series for $\sin(x)$.
- Explicitly multiplying out $\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\right) \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$ yields

$$\left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{5!} \dots\right) + \left(-\frac{x^3}{3!} - \frac{x^4}{3!} - \frac{x^5}{2! \cdot 3!} - \dots\right) + \left(\frac{x^5}{5!} + \frac{x^6}{5!} + \frac{x^7}{2! \cdot 5!} + \dots\right) + \dots$$

and then collecting terms of the same degree yields the start of the series as $x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \dots$.

- Thus, the desired Taylor series up through degree 5 is $\boxed{x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \dots}$.

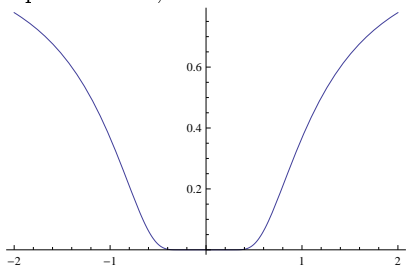
- **Example:** Find the Taylor series for $f(x) = \sin(x) \cos(x)$ at $x = 0$.

- We could find this series by multiplying out the series for $\sin(x)$ and $\cos(x)$.
- However, it is much easier to apply the double-angle identity $\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$, and instead compute the Taylor series for $\sin(2x)$.
- We can obtain the series for $\sin(2x)$ simply by plugging in $2x$ in place of x for the Taylor series of $\sin(x)$.
- This gives the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots$

- Dividing by 2 yields the requested series as $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n+1)!} x^{2n+1} = x - \frac{2^2}{3!} x^3 + \frac{2^4}{5!} x^5 - \frac{2^6}{7!} x^7 + \dots$.

- **Example:** Find the Taylor series for the function $f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$, centered at $x = 0$.

- By the limit definition of derivative, $f'(0) = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x} = \lim_{u \rightarrow \pm\infty} \frac{e^{-u^2}}{1/u} = \lim_{u \rightarrow \pm\infty} \frac{u}{e^{u^2}} = 0$, since the exponential will dominate the polynomial. (We substituted $u = 1/x$ in the middle step.)
- By the Chain Rule, $f'(x) = \frac{e^{-1/x^2}}{x^3}$, so again by the definition, $f''(0) = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}/x^3}{x} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x^4}$. In the same way as before, we can make a substitution to show that this limit is also equal to 0.
- Next, we have $f''(x) = \frac{2 - 3x^2}{x^6} e^{-1/x^2}$, so like before, $f'''(0) = \lim_{x \rightarrow 0} \frac{2 - 3x^2}{x^7} e^{-1/x^2}$, and a slightly lengthier computation eventually shows this limit is also equal to 0.
- In fact, if we continue calculating derivatives of this function at $x = 0$, we will see that they are all equal to zero! Thus, the Taylor series for this function is $0 + 0x + 0x^2 + 0x^3 + 0x^4 + \dots = \boxed{0}$.
- What is happening here is that the function is so flat near the origin that all its derivatives are actually equal to zero, as can be seen from the graph of $y = f(x)$:



8.3 Taylor Polynomials and Convergence of Taylor Series

- We now know how to compute Taylor series, but we have still not answered a very important question: when does an infinite Taylor series actually converge to the original function?
 - Explicitly, we are asking: if $f(x)$ is a function, for which values of x does the power series $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ converge to $f(x)$?
 - We have already seen an example where the Taylor series converges to the function for some but not all values of x – like for $f(x) = \frac{1}{1-x}$, if $|x| < 1$.
 - We have also seen an example where the Taylor series almost never converges to the value of the function: the “flat” function $f(x) = e^{-1/x^2}$ for $x \neq 0$, with $f(0) = 0$, has its Taylor series identically equal to zero, but the function is only zero at $x = 0$.
 - It would be nice if we could tell when we are dealing with a function that converges to its Taylor series, so as to avoid badly-behaved examples like e^{-1/x^2} .
- The partial sums of a Taylor series will be very important in our discussion, so we will give them a name:
- **Definition:** The k th Taylor polynomial for $f(x)$ at $x = a$ is the k th partial sum $T_k(x) = \sum_{n=0}^k \frac{f^{(n)}(a)}{n!} (x-a)^n$ of the Taylor series for $f(x)$.
 - In other words, it is the Taylor series summed “up to the k th power”. Thus for example, $T_2(x) = f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2} (x-a)^2$.
 - **Example:** The degree-4 Taylor polynomial for $f(x) = \cos(x)$ at $x = 0$ is $T_4(x) = \boxed{1 - \frac{x^2}{2!} + \frac{x^4}{4!}}$.
 - Notice that the formula for the linearization of the function $y = f(x)$ at $x = a$ is the linear (degree-1) Taylor polynomial.
 - The degree- k Taylor polynomial agrees with the value of f and its first k derivatives at $x = a$, meaning that $T_k^{(d)}(a) = f^{(d)}(a)$ for $0 \leq d \leq k$.

8.3.1 Taylor's Theorem

- We are interested in the size of the “remainder term” $R_k(x) = f(x) - T_k(x)$ that shows up when we approximate a function $f(x)$ by one of its Taylor polynomials $T_k(x)$.
- **Theorem** (Taylor's Theorem): Suppose $f(x)$ is a function whose $(k + 1)$ st derivative is continuous. If $T_k(x)$ is the k th Taylor polynomial for $f(x)$ at $x = a$, and $R_k(x) = T_k(x) - f(x)$ is the “remainder term”, then for any value b , we have

$$|R_k(b)| \leq M \cdot \frac{|b - a|^{k+1}}{(k + 1)!},$$

where M is any constant such that $|f^{(k+1)}(x)| \leq M$ for all x in the interval $[a, b]$.

- There are a number of different formulas for the remainder term in Taylor's Theorem.
 - * The “integral form” of the remainder estimate is: if $f^{(k+1)}(x)$ is continuous on $[a, b]$, then $R_k(b) = \frac{1}{k!} \int_a^b (b - t)^k f^{(k+1)}(t) dt$.
 - * Notice that the case $k = 0$ of the integral form says that $f(b) - f(a) = \frac{1}{1!} \int_a^b f'(t) dt$, which is the Fundamental Theorem of Calculus.
 - * Lagrange's form of the remainder estimate is: if $f^{(k+1)}(x)$ is continuous on $[a, b]$, then there exists a number ζ in (a, b) such that $R_k(b) = \frac{(b - a)^{k+1}}{(k + 1)!} \cdot f^{(k+1)}(\zeta)$.
 - * Notice that the case $k = 0$ of Lagrange's estimate says that there exists a number ζ in (a, b) such that $f(b) - f(a) = (b - a) \cdot f'(\zeta)$, which is equivalent to the Mean Value Theorem.
- **Proof** ($k = 1$ case with $b > a$): By hypothesis, $|f''(x)| \leq M$ for all x in $[a, b]$, so in particular, $f''(x) \leq M$.
 - * By integrating both sides on the interval $[a, x]$, we see that $\int_a^x f''(t) dt \leq \int_a^x M dt$.
 - * Evaluating the integrals gives $f'(x) - f'(a) \leq M(x - a)$, or equivalently, $f'(x) \leq f'(a) + M(x - a)$.
 - * Integrating both sides on the interval $[a, b]$ then yields $\int_a^b f'(x) dx \leq \int_a^b [f'(a) + M(x - a)] dx$.
 - * Evaluating the integrals gives $f(b) - f(a) \leq (b - a)f'(a) + M \cdot \frac{(b - a)^2}{2!}$.
 - * Rearranging yields $R_k(b) \leq M \cdot \frac{(b - a)^2}{2!}$.
 - * In a similar manner, we can obtain the lower bound $-M \cdot \frac{(b - a)^2}{2!} \leq R_k(b)$ by integrating the inequality $-M \leq f''(x)$. Together, these give the desired bound.
- The proof where $b < a$ is essentially the same (just with the endpoints reversed), and the proof for general k follows in the same way by integrating $k + 1$ times.
- Taylor's Theorem implies that the k th Taylor polynomial $T_k(x)$ gives the best approximation to the function $f(x)$ near $x = a$, among all polynomials of degree k .
 - Explicitly, the theorem implies that $T_k(x)$ is the only polynomial $p(x)$ such that as $x \rightarrow a$, the difference $f(x) - p(x)$ shrinks faster than $(x - a)^k$.

8.3.2 Convergence of Common Taylor Series

- The estimate given by Taylor's Theorem is powerful enough to prove that the Taylor series for a number of common functions will actually converge to the function's value.
 - The idea behind most of these proofs is to prove that the remainder term $R_k(x)$ goes to zero as $k \rightarrow \infty$.
- **Example**: Show that the Taylor series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+1}$ converges to $\sin(x)$ for all x .

- We note that $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ is the Taylor series for $\sin(x)$ at $x = 0$, so we will set $a = 0$.
 - In order to apply Taylor's Theorem, we first have to find a constant M such that $|f^{(k+1)}(x)| \leq M$ for all x in the interval $[0, b]$.
 - Since the derivatives of f are all either $\pm \sin(x)$ or $\pm \cos(x)$, we see that for any k and any x , it is always true that $|f^{(k+1)}(x)| \leq 1$. Therefore, we can take $M = 1$.
 - Then, applying Taylor's Theorem to $f(x) = \sin(x)$ with $a = 0$ yields the bound $|R_k(b)| \leq \frac{|b|^{k+1}}{(k+1)!}$.
 - As $k \rightarrow \infty$, for any value of b , the value of $\frac{|b|^{k+1}}{(k+1)!}$ tends to 0, since the factorial grows faster than any polynomial.
 - Therefore, as $k \rightarrow \infty$, the remainder term tends to zero, meaning that $T(b) = f(b) = \sin(b)$, as claimed.
- Example: Show that the Taylor series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ converges to $\cos(x)$ for all x .
 - From the previous example, we know that $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$. As we saw earlier, the radius of convergence for this power series is $R = \infty$.
 - Differentiating both sides immediately yields that $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ for all x , because we are allowed to differentiate a power series term-by-term within its radius of convergence.
 - Of course, we could also have made a similar argument using Taylor's Theorem like we did for sine, but it was not necessary to duplicate the calculation we already made.
 - Example: Show that the Taylor series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges to e^x for all x .
 - We note that $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ is the Taylor series for e^x at $x = 0$, so we will set $a = 0$.
 - In order to apply Taylor's Theorem, we first have to find a constant M such that $|f^{(k+1)}(x)| \leq M$ for all x in the interval $[0, b]$.
 - Since $f^{(k+1)}(x) = e^x$ for any k , we see that $|f^{(k+1)}(x)| \leq e^b$ on the interval $[0, b]$ if $b > 0$, and $|f^{(k+1)}(x)| \leq 1 \leq e^{-b}$ on the interval $[b, 0]$ if $b < 0$. To cover both cases at once, we can take $M = e^{|b|}$.
 - Then, applying Taylor's Theorem to $f(x) = e^x$ with $a = 0$ yields the bound $|R_k(b)| \leq e^{|b|} \frac{|b|^{k+1}}{(k+1)!}$.
 - As $k \rightarrow \infty$, for any value of b , the value of $e^{|b|} \frac{|b|^{k+1}}{(k+1)!}$ tends to 0, since the factorial grows faster than any polynomial. (Note that $e^{|b|}$ does not depend on k , so it acts like a constant.)
 - Therefore, as $k \rightarrow \infty$, the remainder term tends to zero, meaning that $T(b) = f(b) = e^b$, as claimed.
 - Note in particular that if we set $x = 1$ in the Taylor series expansion, we obtain the interesting fact that $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$.
 - Example: Show that, for any k , the binomial series $\sum_{n=0}^{\infty} \binom{k}{n} x^n$ converges to $(1+x)^k$ whenever $|x| < 1$.
 - Although it is possible to show this result using Taylor's Theorem to bound the size of the remainder term, it is very difficult. (The problem is that it is not so easy to get a sufficiently good estimate on the magnitude of the $n+1$ st derivative as $n \rightarrow \infty$.)

- We will instead use a different method: let $B(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \dots$.
- First, observe that for $a_n = \binom{k}{n} x^n$ we have $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{k-n}{n+1} x \right| = |x|$, so by the Ratio Test the power series converges for $|x| < 1$.
- Next, differentiate $B(x)$ term-by-term to see that $B'(x) = k + (k-1)x + \frac{k(k-1)(k-2)}{2!}x^2 + \dots$.
- Then, by multiplying out and collecting terms, we see that

$$(1+x)B'(x) = (1+x) \cdot \left(k + (k-1)x + \frac{k(k-1)(k-2)}{2!}x^2 + \dots \right) = k + k^2x + \frac{k^2(k-1)}{2!}x^2 + \dots = k \cdot B(x).$$
- Now notice that the derivative of $(1+x)^{-k}B(x)$ is $-k(1+x)^{-k-1}B(x) + (1+x)^{-k}B'(x) = \frac{(1+x)B'(x) - kB(x)}{(1+x)^{k+1}} = 0$.
- Therefore, the derivative of $(1+x)^{-k}B(x)$ is identically zero, meaning that $(1+x)^{-k}B(x)$ is a constant function for $|x| < 1$.
- Since $B(0) = 1$, we conclude that $B(x) = (1+x)^k$ whenever $|x| < 1$.

8.3.3 Table of Common Taylor Series

- For reference, here is a table of the most commonly used Taylor series:

Function	Series	Initial Terms	Converges for
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + x^4 + \dots$	$-1 < x < 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$	All x
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	All x
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	All x
$\tan^{-1}(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$	$-1 \leq x \leq 1$
$\ln(1+x)$	$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$	$-1 < x \leq 1$
$(1+x)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n$	$1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$	$-1 < x < 1$

- We have centered each of these series at $x = 0$, since the terms are easiest to write down there.
 - We have already found the Taylor series for all of the functions listed above. We omit the details of the verifications that the remaining Taylor series converge to the function values.

8.4 Applications of Taylor Series

- Taylor series are used extensively in the physical sciences owing to their ability to closely approximate complicated functions with simpler ones (namely, polynomials). This is especially true in physics, where it is very common to end up with functions that are too hard to analyze exactly. In order to be able to analyze the relevant behavior, a common procedure is to use a Taylor polynomial of small degree to get an approximate result.

- As an example, Planck's law says that the spectral radiance of a black body at frequency ν and temperature T is given by the formula $B(\nu, T) = \frac{2h\nu^3}{c^2} \cdot \left(e^{\frac{h\nu}{kT}} - 1 \right)^{-1}$, where k is Boltzmann's constant, h is Planck's constant, and c is the speed of light.
- It is of interest to understand the behavior of this function when $\nu \rightarrow 0$ (the "low-frequency limit"), and when $\nu \rightarrow \infty$ (the "high-frequency limit") since previous models were unable to capture both behaviors correctly.
- By expanding the function $B(\nu, T)$ as a Taylor series in ν at $\nu = 0$, it is easy to analyze the behavior as $\nu \rightarrow 0$: one obtains $B(\nu, T) \approx \frac{2kT}{c^2} \nu^2$, a result known as the Rayleigh-Jeans law.
- In a similar way, one sees that for large ν , it is true that $B(\nu, T) \approx \frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{kT}}$, a result known as the Wein approximation.

8.4.1 Summing Series Using Taylor Expansions

- Taylor's Theorem gives us a new way to evaluate infinite series: if we recognize that a series is actually a Taylor series of some function evaluated at some point, then (provided the Taylor series converges to the value of the function) the value of the function is the sum of the series.
 - Even if the series is not absolutely convergent at the given point, the error estimate from Taylor's Theorem might be strong enough to allow us to determine the value of the series anyway.
- Example: Find the Taylor series for $\ln(1+x)$ at $x=0$, and then write down an infinite series for $\ln(2)$.

- We have $f'(x) = \frac{1}{1+x}$, $f''(x) = \frac{-1}{(1+x)^2}$, $f'''(x) = \frac{2}{(1+x)^3}$, $f^{(4)}(x) = \frac{-3!}{(1+x)^4}$, and in general, $f^{(n)}(x) = \frac{(-1)^{n-1} \cdot (n-1)!}{(1+x)^n}$, meaning that $f^{(n)}(0) = (-1)^{n-1} \cdot (n-1)!$ for $n \geq 1$.

- Thus, the Taylor series for $\ln(1+x)$ at $x=0$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$.

- Now we will use Taylor's Theorem, with $f(x) = \ln(1+x)$, $a=0$, and $b=1$.
- On the interval $[0, 1]$, the function $f^{(n+1)}(x)$ takes its maximum absolute value at $x=0$, and the maximum is $(-1)^n \cdot n!$. Thus, we can take $M = n!$, and then Taylor's Theorem implies that

$$|T_k(1) - \ln(2)| \leq \frac{M \cdot |b-a|^{n+1}}{(n+1)!} = \frac{n! \cdot 1^{n+1}}{(n+1)!} = \frac{1}{n+1}.$$

- In other words, the partial sum $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots + \frac{(-1)^k}{k}$, is within $\frac{1}{k+1}$ of $\ln(2)$.

* For example, if we take $k=99$, our result says that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{99} = 0.6982$ is within 0.01 of $\ln(2) = 0.6931$, which indeed it is.

- Taking the limit as $k \rightarrow \infty$ provides a proof that $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$.

(Notice that this is simply the alternating harmonic series.)

- Note that the Alternating Series Test can be applied to show that the alternating harmonic series converges (and it gives the same error bound on partial sums), but it does not say anything about the actual value.
- Taylor's Theorem, then, has given us a very nontrivial piece of new information: namely, the actual value of the alternating harmonic series.

- Example: Show that the series $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges, and then find its sum.
 - It is an easy matter to apply the Ratio Test to see that this series converges, since, if $a_n = \frac{2^n}{n!}$, then $\frac{a_{n+1}}{a_n} = \frac{2}{n+1}$, which goes to 0 as $n \rightarrow \infty$. However, the Ratio Test (or any other convergence test) will shed no light on the actual value of the series.
 - To find the value, we try to recognize this series as the value of a Taylor series: in this case, we see that it is the Taylor series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ evaluated at $x = 2$.
 - We know that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is the Taylor series for e^x , and that this series converges to e^x for all x . We then set $x = 2$ to conclude that $\sum_{n=0}^{\infty} \frac{2^n}{n!} = \boxed{e^2}$.
- Example: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$ converges, and then find its sum.
 - It is an easy matter to apply the Comparison Test to see that this series converges, since $\frac{1}{n \cdot 2^n} \leq \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.
 - To find the value, we try to recognize this series as the value of a Taylor series: in this case, we see that it is the Taylor series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ evaluated at $x = \frac{1}{2}$.
 - Now notice that the derivative of $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, which converges for $|x| < 1$.
 - Then for some C , we have $\sum_{n=1}^{\infty} \frac{x^n}{n} = \int \frac{1}{1-x} dx = -\ln(1-x) + C$, valid for $|x| < 1$. To find C , we simply set $x = 0$ to obtain $C = 0$.
 - Hence, $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is the Taylor series for $f(x) = -\ln(1-x)$, convergent for $|x| < 1$.
 - So we can set $x = 1/2$ to obtain $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} = -\ln(1/2) = \boxed{\ln(2)}$.

8.4.2 Numerical Approximations Via Taylor Series

- The error estimate from Taylor's Theorem allows us to make arbitrarily good estimates of function values using a power series expansion.
 - Exponentials, logarithms, radicals, and trigonometric functions are generally impossible to compute by hand directly. (For example, what is $\ln(3.22)$?)
 - However, if we replace such a function with its Taylor series (centered at a nearby location to make computation easy), then we can use the Taylor expansion to compute the value of the function, within a bounded error tolerance.
 - This is, almost exactly, how computing devices actually evaluate expressions like $e^{1.402}$: they use a small amount of stored data (like $e^1 = 2.71828\dots$) combined with the necessary series expansions and error estimates to find values between their stored data points.

- Example: Calculate the value of $\sqrt{1.01}$ to six decimal places, by hand.
 - The Taylor series for \sqrt{x} at $x = 1$ is $1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 + \dots$, by repeated differentiation.
 - * This is also an example of the binomial expansion, applied to $(1+x)^{1/2}$.
 - Therefore, for $f(x) = \sqrt{x}$, $a = 1$, $b = 1.01$, and $k = 2$, we see that on the interval $[1, 1.01]$, the function $f'''(x) = \frac{3}{8}x^{-5/2}$ is bounded above by its value at 1: namely, $\frac{3}{8}$.
 - Then Taylor's Theorem tells us that $|T_2(1.01) - \sqrt{1.01}| \leq \frac{3}{8} \cdot \frac{|1.01 - 1|^3}{3!} = \frac{10^{-6}}{16} < 10^{-7}$.
 - But $T_2(1.01) = 1 + \frac{1}{2}(0.01) - \frac{1}{8}(0.01)^2 = 1 + 0.005 - 0.000125 = 1.0049875$.
 - Because we know that the error is less than 10^{-7} , we conclude that $\boxed{\sqrt{1.01} \approx 1.004988}$ to six decimal places.

- Example: Calculate the value of $e^{0.03}$ to eight decimal places.
 - The Taylor series for e^x at $x = 0$ is $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$.
 - We want to determine the number of terms we will need to estimate this value. Since $f^{(n)}(x) = e^x$, we see that on the interval $[0, 0.03]$, the maximum value of $f^{(n)}(x)$ is just $e^{0.03} < e < 3$. Thus, we can take $M = 3$.
 - Now we take $f(x) = e^x$, $a = 0$, $b = 0.03$, and $M = 3$ in Taylor's Theorem: this yields the error bound

$$|T_n(e^{0.03}) - e^{0.03}| \leq 3 \frac{|0.03 - 0|^{n+1}}{(n+1)!} = 3 \cdot \frac{0.03^{n+1}}{(n+1)!}.$$

- If we try $n = 4$, the upper bound is $3 \cdot \frac{0.03^5}{5!} = \frac{3^6}{120 \cdot 10^{10}} < \frac{10^3}{10^{12}} = 10^{-9}$, so it will be accurate to at least 8 decimal places.
- The desired approximation is $T_4(0.03) = 1 + 0.03 + \frac{0.03^2}{2} + \frac{0.03^3}{6} + \frac{0.03^4}{24} = 1.03045453375$, which, to eight decimal places, gives the approximation $\boxed{e^{0.03} \approx 1.03045453}$.

8.4.3 Approximating Functions by Polynomials

- On the most basic level, the fact that a function possesses a convergent Taylor series says that, on the interval of convergence, the function can be approximated arbitrarily closely by a polynomial: namely, an appropriate Taylor polynomial.
- For example, the fact that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all real x means that the sequence of polynomials $1, 1 + x, 1 + x + \frac{x^2}{2!}, 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}, \dots$ will eventually approximate e^x within any arbitrarily small accuracy, for any fixed x .
 - Moreover, we can obtain an upper bound on the size of the error on any interval $[a, b]$ using Taylor's Theorem.
 - Having such a bound is useful because it allows us to write down uniform approximation to the function e^x on an entire interval, rather than merely finding an approximate value at a single point.
- Example: Approximate the function $f(x) = \sqrt{x}$ by a polynomial of degree 3 near $x = 4$. Then bound the maximum error of the approximation on the interval $[3, 5]$.
 - We will use the degree-3 Taylor polynomial for \sqrt{x} at $x = 4$.

- To find the coefficients, we can compute $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, $f'''(x) = \frac{3}{8}x^{-5/2}$, so that $f'(4) = \frac{1}{4}$, $f''(4) = -\frac{1}{32}$, and $f'''(4) = \frac{3}{256}$.
- Thus, the desired Taylor polynomial is $\sqrt{x} \approx T_3(x) = \boxed{2 + \frac{1}{4}(x-4) - \frac{1}{32}(x-4)^2 + \frac{3}{256}(x-4)^3}$.
- To bound the error, we use Taylor's Theorem with $f(x) = \sqrt{x}$ and $a = 4$.
- We need to find an M such that $|f''''(x)| \leq M$ for all x in $[3, 5]$. Since $f''''(x) = -\frac{15}{16}x^{-7/2}$, the maximum value occurs when $x = 3$, and so we can take $M = \frac{15}{16} \cdot 3^{-7/2}$.
- Then, for any b in the interval $[3, 5]$, we have $|b-4| \leq 1$, so Taylor's Theorem gives $|R_3(x)| \leq M \cdot \frac{|b-4|^4}{4!} \leq \frac{M}{4!} < \boxed{8.35 \cdot 10^{-4}}$.
- **Example:** Approximate the function $f(x) = \sin(x)$ by a polynomial on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that the error is at most 0.01 at any point in the interval.

- We will use a Taylor polynomial for $\sin(x)$ at $x = 0$: what we want is to determine how big we need to take the degree to get the desired accuracy.
- The Taylor series for $f(x) = \sin(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$, and we know it converges to $\sin(x)$ for all real x .
- In order to apply Taylor's Theorem, we first need to find a constant M such that $|f^{(k+1)}(x)| \leq M$ for all x in the interval $[-\pi, \pi]$. As we saw when we proved that the Taylor series for sine converges to $\sin(x)$, we can take $M = 1$ since each derivative is $\pm \sin(x)$ or $\pm \cos(x)$.
- Applying Taylor's Theorem to $f(x) = \sin(x)$ with $a = 0$ shows that

$$|R_k(b)| \leq \frac{|b|^{k+1}}{(k+1)!} \leq \frac{(\pi/2)^{k+1}}{(k+1)!}$$

for any b in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, since $|b| \leq \frac{\pi}{2}$ for any such b .

- We want to ensure that this quantity is at most 0.01. It can be checked with a calculator that $\frac{(\pi/2)^6}{6!} \approx 0.00468$, so we may take $k = 6$.
- The desired polynomial is then $T_6(x) = \boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!}}$. At $x = \pi/2$, for example, its value is 1.0045, as compared to $\sin(\pi/2) = 1$.

8.4.4 Computing Integrals Using Series Expansions

- Unlike with differentiation, it is not always possible to compute a closed-form expression for arbitrary antiderivatives of elementary functions in terms of other elementary functions.
 - An elementary function is any combination of algebraic, trigonometric, or exponential functions (or their inverses): thus, for example, $e^{\sin(\sqrt{x})}$ and $\tan^{-1}(\ln(1-3x^3))$ are elementary functions.
 - Also, even when a function does have an elementary antiderivative, it can sometimes be much more complicated than the original function. For example,

$$\int \sqrt{\tan(x)} dx = \frac{1}{\sqrt{8}} \left[2 \tan^{-1} \left(\frac{\sqrt{2 \tan(x)}}{1 - \tan(x)} \right) + \ln \left(\frac{1 - \sqrt{2 \tan(x)} + \tan(x)}{1 + \sqrt{2 \tan(x)} + \tan(x)} \right) \right] + C.$$

- If $f(x)$ is a function that has a Taylor series expansion, however, we can simply integrate term-by-term to find its antiderivative. We will then usually be able to find an easy upper bound on the error, thus allowing us to compute an approximation of the integral within any desired accuracy.

- It is also worth noting that, unlike other numerical integration procedures like computing a Riemann sum or using Simpson's Rule, using a Taylor series only requires us to evaluate polynomials (as opposed to evaluating many general function values).

- **Example:** Find a series expansion for a function whose derivative is e^{-x^2} . Use the result to estimate the value of $\int_0^1 e^{-x^2} dx$ to two decimal places.

- The Taylor series for e^x is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so to get a series for e^{-x^2} we simply plug in $-x^2$ to obtain

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$$

- Integrating term-by-term gives the general antiderivative

$$\int e^{-x^2} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot (2n+1)} x^{2n+1} = C + x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots$$

- We can set $C = 0$ to get a specific antiderivative $F(x) = \boxed{x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots}$.

- To estimate the integral, we know by the Fundamental Theorem of Calculus that $\int_0^1 e^{-x^2} dx = F(1) - F(0) = F(1) = 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} - \frac{1}{11 \cdot 5!} + \dots$.

- Since this is an alternating series whose terms are decreasing in magnitude, we know that the size of the error is bounded by the next term of the series.

- Therefore, the estimate $1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} \approx 0.738$ will be within $\frac{1}{11 \cdot 5!} < \frac{1}{1000}$ of the actual value of the sum, and it will be an underestimate.

* Another approach for bounding the error would be to use Taylor's Remainder Theorem, but this is trickier because it requires having an upper bound on the derivatives of e^{-x^2} .

- Thus, to two decimal places, the value of the integral is $\boxed{0.74}$.

- **Example:** Find a series expansion for the value of the integral $\int_0^1 \sqrt{1+x^3} dx$. Use the result to estimate the value of the integral to two decimal places.

- First, we know that the binomial series expansion for $\sqrt{1+x}$ is $\sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$,

$$\text{so the expansion for } \sqrt{1+x^3} \text{ is } \sum_{n=0}^{\infty} \binom{1/2}{n} x^{3n} = 1 + \frac{1}{2}x^3 - \frac{1}{8}x^6 + \frac{1}{16}x^9 - \dots,$$

- Integrating term-by-term shows that an antiderivative is $F(x) = \sum_{n=0}^{\infty} \frac{1}{3n+1} \cdot \binom{1/2}{n} x^{3n+1} = x + \frac{1}{8}x^4 - \frac{1}{56}x^7 + \frac{1}{160}x^{10} - \dots$.

- Then $\int_0^1 \sqrt{1+x^3} dx = F(1) - F(0) = F(1) = \boxed{\sum_{n=0}^{\infty} \frac{1}{(3n+1)} \cdot \binom{1/2}{n}}$.

- The first few terms in the series are $1 + \frac{1}{8} - \frac{1}{56} + \frac{1}{160} - \frac{5}{1664} + \dots$.

- After the first two terms, it is easy to see that this series is alternating and that the terms are decreasing in magnitude, so by the Alternating Series Test, the estimate $1 + \frac{1}{8} - \frac{1}{56} + \frac{1}{160} \approx 1.1134$ is within $\frac{5}{1664} < \frac{1}{300}$ of the actual value, and it is an overestimate

- Hence, to two decimal places, the value of the integral is $\boxed{1.11}$.

8.4.5 Computing Limits Using Series Expansions

- Using series expansions, we can compute limits.
 - The basic idea for using Taylor series to compute a limit of the form $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is to find Taylor series expansions for both f and g at $x = a$, and then compare the relevant lowest-degree terms. (We can also evaluate other limit forms as well in this manner, but a quotient is the most common.)
 - Although in principle many such limits can also be handled with enough applications of L'Hôpital's Rule, many computer algebra systems actually use Taylor series expansions to compute limits because it is faster to compute series expansions.
 - Furthermore, applying L'Hôpital's Rule many times can rapidly become very cumbersome if the functions involved are at all complicated.
- Example: Find $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{\cos(x) - 1}$.
 - Since the Taylor series for e^x is $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$, the Taylor series for $e^x - x - 1$ is $\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$.
 - Likewise, since the Taylor series for $\cos(x)$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$, the Taylor series for $\cos(x) - 1$ is $-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$.
 - Therefore, $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{\cos(x) - 1} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots}{-\frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots} = \lim_{x \rightarrow 0} \frac{\frac{1}{2} + \frac{1}{6}x + \frac{1}{24}x^2 + \dots}{-\frac{1}{2} + \frac{1}{24}x^2 - \frac{1}{720}x^4 + \dots} = \boxed{-1}$, where we simply set $x = 0$ in the numerator and denominator at the last step.
 - Note that these manipulations are valid because all of the power series involved are continuous at $x = 0$.
 - We can also find the limit using L'Hôpital's Rule twice: $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{\cos(x) - 1} = \lim_{x \rightarrow 0} \frac{e^x - 1}{-\sin(x)} = \lim_{x \rightarrow 0} \frac{e^x}{-\cos(x)} = \boxed{-1}$.
- Example: Find $\lim_{x \rightarrow 0} \frac{x^2 \sin(2x^3)}{\tan^{-1}(x^5)}$.
 - Since the Taylor series for $\sin(x)$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$, the Taylor series for $x^2 \sin(2x^3)$ is given by the expression $x^2 \cdot \left[2x^3 - \frac{(2x^3)^3}{3!} + \frac{(2x^3)^5}{5!} - \dots \right] = 2x^5 - \frac{2^3}{3!}x^{11} + \frac{2^5}{5!}x^{17} - \dots$.
 - Similarly, since the Taylor series for $\tan^{-1}(x)$ is $x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$, the Taylor series for $\tan^{-1}(x^5)$ is $x^5 - \frac{x^{15}}{3} + \frac{x^{25}}{5} - \dots$.
 - Therefore, $\lim_{x \rightarrow 0} \frac{x^2 \sin(2x^3)}{\tan^{-1}(x^5)} = \lim_{x \rightarrow 0} \frac{2x^5 - \frac{4}{3}x^{11} + \frac{4}{15}x^{17} - \dots}{x^5 - \frac{1}{3}x^{15} + \frac{1}{5}x^{25} - \dots} = \lim_{x \rightarrow 0} \frac{2 - \frac{4}{3}x^6 + \frac{4}{15}x^{11} - \dots}{1 - \frac{1}{3}x^{10} + \frac{1}{5}x^{15} - \dots} = \boxed{2}$, where we simply set $x = 0$ in the numerator and denominator at the last step.
 - Note that all of these manipulations are valid because all of the power series involved are continuous at $x = 0$.
 - It is of course possible to compute this limit using five applications of L'Hôpital's Rule, but the algebra involved in computing the required fifth derivatives is quite extensive (and unpleasant to do without a computer).

8.4.6 Euler's Formula

- As our last example, we will show how Euler's Formula can be found very naturally using Taylor series.
- We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for any real number x . In fact, this formula also holds for any complex number x .
- Although we will omit the (rather lengthy) details, all of our results on convergence of series can be extended to series of complex numbers as well.
- Setting $x = i\theta$ then produces

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right). \end{aligned}$$

- But now notice that the real part is the Taylor series for $\cos(\theta)$, while the imaginary part is the Taylor series for $\sin(\theta)$.
- Thus, we obtain $e^{i\theta} = \cos(\theta) + i \sin(\theta)$, which is precisely Euler's formula.

Well, you're at the end of my handout. Hope it was helpful.

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