Contents

6 Parametric Curves, Polar Coordinates, and Complex Numbers

In this chapter, we introduce the notion of a parametric curve, and then analyze a number of related calculus problems: finding the tangent line to, the arclength of, and the area underneath a parametric curve.

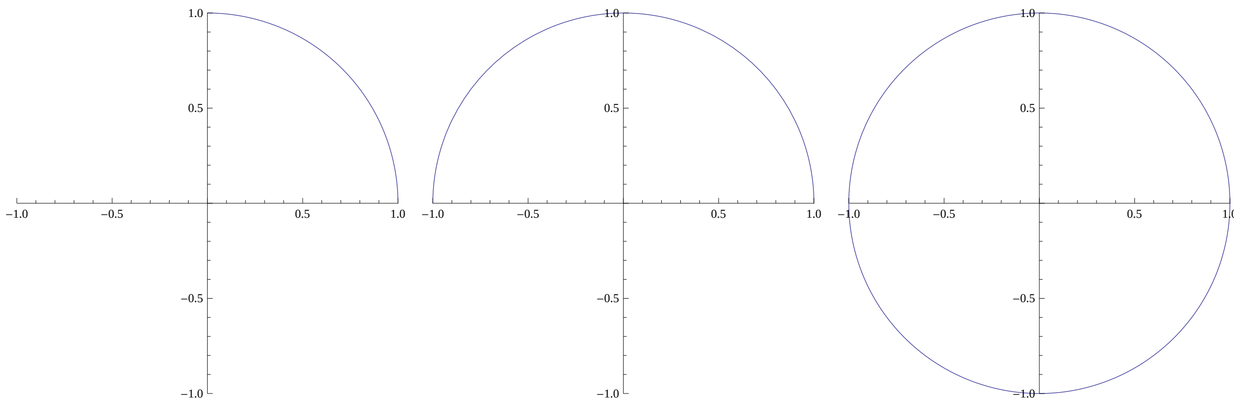
We then introduce a new coordinate system called polar coordinates (which often shows up in physical applications) and analyze polar graphing. We then discuss calculus in polar coordinates, and solve the tangent line, arclength, and area problems for polar curves.

6.1 Parametric Equations and Parametric Curves

- Up until now, we have primarily worked with single functions of one variable: something like $y = f(x)$, where $f(x)$ expresses y explicitly as a function of the variable x.
  - We have also occasionally discussed functions defined implicitly by some kind of relation between y and x.
  - Often in physical applications, we will have systems involving a number of variables that are each a function of time, and we want to relate those variables to each other.
  - One very common situation is the following: a particle moves around in the Cartesian plane (i.e., the xy-plane) over time, so that both its x-coordinate and y-coordinate are functions of t.
- Definition: If $x(t)$ and $y(t)$ are functions of t, then the set of points $(x, y) = (x(t),\, y(t))$ is called a parametric curve. The points on this curve are traced out as t varies over the real numbers.
  - Given a parametrization $x = x(t)$ and $y = y(t)$, we would like to analyze properties of the "parametric curve" that $(x(t), y(t))$ traces out in the plane, as t varies over some interval.
  - To graph a parametric curve, it is sometimes possible to find a Cartesian equation for the curve of the form $y = f(x)$. However, there is no general method for doing this.
  - In lieu of some sort of clever way to eliminate the variable t, the standard method is just to plug in many values of the parameter t to find some points $(x(t), y(t))$ on the curve, and then connect them up with a smooth curve.
- Example: Sketch the parametric curve given by $x = \cos(t)$, $y = \sin(t)$, for $0 \le t \le 2\pi$.
  - To sketch a parametric curve, we first make a table of values: we plug in easy values for t and compute the corresponding points (x, y).

t	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
x	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1
y	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

- If we plot these points and then join them up with a smooth curve, we see that resulting shape seems to be a circle.
- Indeed, it is a circle: from the Pythagorean identity $\sin^2(t) + \cos^2(t) = 1$, we immediately see that if $x = \cos(t)$ and $y = \sin(t)$, then $x^2 + y^2 = 1$.
- Graphs of the parametric curve $x = \cos(t)$, $y = \sin(t)$ for $0 \le t \le \pi/2$, $0 \le t \le \pi$, and $0 \le t \le 2\pi$ are below:



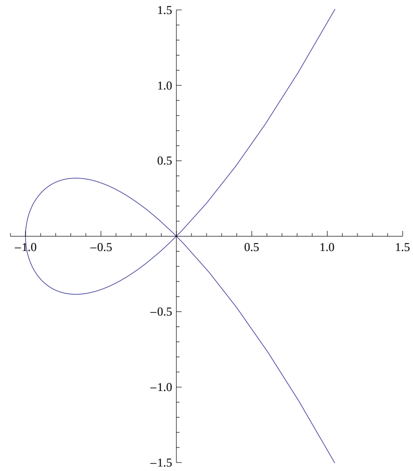
- Example: Describe the parametric curve $x = \cos(2t)$, $y = \sin(2t)$, for $0 \le t \le 2\pi$.
  - Motivated by the previous example, we see that the Pythagorean identity $\sin^2(2t) + \cos^2(2t) = 1$ again implies that $x^2 + y^2 = 1$, so this curve is once again the unit circle.
  - However, the parametrization is different: if we plug in a few values for t, we will in general not end up with the same points as before. In fact, it is easy to see that this parametrization will move along the unit circle twice as quickly as the previous one.
  - To emphasize: any curve in the plane has many different parametrizations!

- Example: Describe the parametric curve $x = a\cos(t)$, $y = b\sin(t)$, for $0 \le t \le 2\pi$, where a and b are positive real numbers.
  - From our analysis earlier, this curve will have the same general shape as a circle, but stretched by a factor of a in the x-direction and by a factor of b in the y-direction.
  - This describes an ellipse: specifically, the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. (That this is a Cartesian equation for the parametric curve is easy to see, once it is pointed out.)

- Example: Sketch the parametric curve $x = t^2 - 1$, $y = t^3 - t$ for $-\infty < t < \infty$.
  - We make a short table of values:

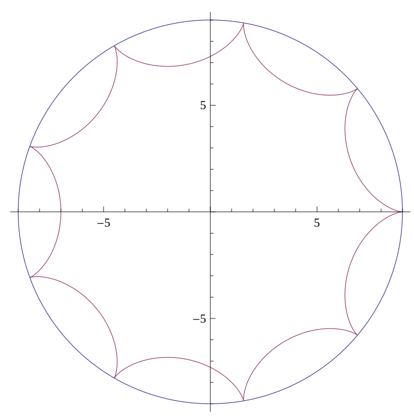
t	-2	-1	$-2/3$	$-1/2$	$-1/3$	0	$1/3$	$1/2$	$2/3$	1	2
x	3	0	$-5/9$	$-3/4$	$-8/9$	-1	$-8/9$	$-3/4$	$-5/9$	0	3
y	-6	0	$10/27$	$3/8$	$8/27$	0	$-8/27$	$-3/8$	$-10/27$	0	6

  - Using the table, we can sketch the graph:



- Notice that this curve crosses itself at the origin: it passes through the origin both when $t = -1$ and when $t = 1$.
- This curve has a special name: it is called a (singular) elliptic curve, plotted above,
- It can be verified that this particular curve also has a Cartesian equation given by $y^2 = x^3 + x^2$.
- An elliptic curve is a curve having the general form $y^2 = x^3 + ax^2 + bx + c$; such curves have a surprisingly wide variety of applications, including in cryptography and in factoring algorithms.

- Example: Sketch the parametric curve described by $x = 8\cos(t) + \cos(8t)$, $y = 8\sin(t) - \sin(8t)$, for $0 \le t \le 2\pi$.
  - It is not particularly easy (or worthwhile) to plug in enough points to create an accurate picture by hand. Instead, it is best to use a computer; here is the result:



  - The resulting graph is a nine-pointed hypocycloid. (A hypocycloid is a curve traced out by a point on a circle being rolled around the inside of a larger circle.) The graph of this hypocycloid – along with the bounding circle, which has radius 9 – is given below (on the left).

- Example: Describe the parametric curve $x = t$, $y = 0$ for $-\infty < t < \infty$, and compare it to the curve $x = \tan(t)$, $y = 0$ for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$.
  - The first curve is just the y-axis, where the particle tracing out the curve moves at constant speed.
  - The second curve is also the y-axis, but this time the particle moves at non-constant speed. (In fact, it takes only a finite amount of time to cover the entire axis.)

6.2 Tangent Lines, Arclengths, and Areas for Parametric Curves

- Once we know what a curve looks like, the next natural questions we tend to ask (this is after all a calculus class) are the following:

1. How can we find the tangent line to the curve?
2. How can we find the arclength of the curve?
3. How can we find the area underneath (or inside, around, described by...) the curve?

- Here are the answers to these three questions for parametric curves:

- Theorem: The slope $\frac{dy}{dx}$ of the tangent line to the curve $(x(t), y(t))$ is given by $\boxed{\frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}}$.

    - Proof: Over a small time interval Δt, we have $\frac{\Delta y}{\Delta x} = \frac{\Delta y/\Delta t}{\Delta x/\Delta t}$.
        - Taking the limit as $\Delta t \to 0$ we see that $\Delta x \to 0$ also, and so $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$, while $\frac{\Delta y}{\Delta t} \to \frac{dy}{dt}$ and $\frac{\Delta x}{\Delta t} \to \frac{dx}{dt}$. [This is just the limit definition of the derivative, three times.]
        - But this is exactly the statement of the theorem.
    - We retain the same interpretation of the value of $\frac{dy}{dx}$ as for an explicit curve of the form $y = f(x)$: if $\frac{dy}{dx}$ is positive, then the curve is increasing, while if $\frac{dy}{dx}$ is negative then the curve is decreasing.
    - For parametric curves, in particular, the tangent line is horizontal when the slope is zero, which (from the expression above) typically occurs when $y'(t) = 0$.
    - It is also possible for the tangent line to be vertical: this will occur when the slope blows up to ∞ or $-\infty$, which typically occurs when $x'(t) = 0$.
    - We can also compute higher-order derivatives using the same procedure as above: for example, the second derivative is $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt} = \frac{y''x' - y'x''}{(x')^3}$, where we computed the derivative of the numerator using the Quotient Rule. Again, this has the same interpretation as with a function of one variable: the second derivative is positive when the curve is concave up, and it is negative when the curve is concave down.

- Theorem: The arclength s of the curve $(x(t), y(t))$ for $t_1 \le t \le t_2$ is given by the integral $\boxed{s = \int_{t_1}^{t_2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt}$.

    - Important Warning: When simplifying the square root, do not forget that $\sqrt{x^2} = |x|$, not x. You should NEVER end up with a negative number as an arclength.
    - Proof: The Pythagorean Theorem says that the bit of distance Δs traveled by a particle which moves Δx in the x-direction and Δy in the y-direction satisfies $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$.
        - Dividing by $(\Delta t)^2$ gives $\left(\frac{\Delta s}{\Delta t}\right)^2 = \left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2$, and now taking the limit as $\Delta t \to 0$ gives us the relation $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$. [This is just the limit definition of the derivative, applied three times.]
        - Taking the square root gives $\frac{ds}{dt} = \sqrt{[x'(t)]^2 + [y'(t)]^2}$. Then integrating both sides with respect to t from t_1 to t_2 yields the stated result.

- Theorem: The area A underneath the curve $(x(t), y(t))$ for $t_1 \le t \le t_2$ is given by the integral $\boxed{A = -\int_{t_1}^{t_2} y(t) \cdot x'(t)\, dt}$, provided the curve is traveling counterclockwise as t increases, and does not cross itself or reverse direction.

    - Important Warning: Be exceedingly careful if the curve changes direction as it is being traced out, or if the curve crosses over itself. (Otherwise, the area may be off by a factor of -1.) It may be necessary to break up the region of integration to ensure the correct answer is found.

* If the curve is moving from right to left, the negative sign is necessary. If the curve is moving from left to right, there is no negative sign.

- Useful Note: If the curve is closed – i.e., $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$ – then the area 'under' the curve means the same thing as the area inside the curve.

  * Neat Fact: If the curve is closed, then the area is also given by the formulas $\boxed{A = \int_{t_1}^{t_2} x(t) \cdot y'(t)\,dt}$ or $\boxed{A = \int_{t_1}^{t_2} \frac{1}{2}\left[x(t) \cdot y'(t) - x'(t) \cdot y(t)\right]\,dt}$.

    · Reason: Integration by parts says that $\int_{t_1}^{t_2} x(t) \cdot y'(t)\,dt = x(t) \cdot y(t)\,|_{t_1}^{t_2} - \int_{t_1}^{t_2} y(t) \cdot x'(t)\,dt$. But now the 'extra term' $x(t) \cdot y(t)\,|_{t_1}^{t_2} = x(t_2)y(t_2) - x(t_1)y(t_1)$ is zero, precisely by the assumption that the curve was closed!

    · The third formula follows by adding the first two formulas together and then dividing by 2.

  * Remark: This is a special case of Green's Theorem, which is a result of multivariable calculus.

- Proof (of Theorem): If the curve does not cross itself, then we can think of it as being given implicitly as $y = f(x)$ for some function f.

  * Then the area would be given by $A = \int_{x(t_2)}^{x(t_1)} f(x)\,dx$. Note that the limits of integration are reversed because the curve is traveling counterclockwise, and so a smaller value of t corresponds to a larger value of x.

  * In this integral, if we now make the substitution $x = x(t)$ to get an integral in the variable t, with $dx = x'(t)\,dt$, we obtain $A = \int_{t_2}^{t_1} y(t)\,x'(t)\,dt$. Swapping the limits of integration gives the area formula in the Theorem.

- Example: For the circle $x = \cos(t)$, $y = \sin(t)$, (i) find an equation for the tangent line at time $t = \pi/6$, (ii) find all points on the curve where the tangent line is horizontal or vertical, (iii) find the arclength of the curve on the interval $0 \le t \le 2\pi$, and (iv) find the area enclosed by the curve.

  - We easily compute $x'(t) = -\sin(t)$ and $y'(t) = \cos(t)$.

  - By the first formula, the slope of the tangent line to the circle at $(\cos(t), \sin(t))$ is $-\frac{\cos(t)}{\sin(t)}$.

    * When $t = \pi/6$, the slope of the tangent line is $-\sqrt{3}$. The tangent line also passes through the point $(\cos(t), \sin(t)) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, so it has equation $\boxed{y - \frac{1}{2} = -\sqrt{3} \cdot \left(x - \frac{\sqrt{3}}{2}\right)}$.

    * The expression for the slope is zero when $\cos(t) = 0$, which occurs for $t = \pi/2$, $3\pi/2$, $5\pi/2$, ..., which gives rise to two points on the curve where there is a horizontal tangent line: $\boxed{(0, 1) \text{ and } (0, -1)}$.

    * Similarly, the slope will be undefined when $\sin(t) = 0$, which occurs for $t = 0$, π, 2π, ..., which also gives rise to two points on the curve with a vertical tangent line: $\boxed{(1, 0) \text{ and } (-1, 0)}$.

    * Of course, we didn't need to use the parametrization to find these points: we did it mostly to build confidence that the results are right.

  - By the second formula, the arclength of the circle is $\int_0^{2\pi} \sqrt{[-\sin(t)]^2 + [\cos(t)]^2}\,dt = \int_0^{2\pi} 1dt = \boxed{2\pi}$. Again, we knew this already.

  - By the third formula, the area of the circle is $-\int_0^{2\pi} \sin(t) \cdot (-\sin(t))\,dt = \int_0^{2\pi} \sin^2(t)\,dt = \int_0^{2\pi} \frac{1 - \cos(2t)}{2}\,dt = \boxed{\pi}$. Once again, we knew this, but it's nice to see that the formula gives the right answer.

- Example: For the ellipse $x = 2\cos(t)$, $y = 3\sin(t)$, find (i) an equation for the tangent line to the curve at time $t = \pi/4$, (ii) find all points where the tangent line to the curve is horizontal or vertical, (iii) set up an integral for the arclength of the curve on the interval $0 \le t \le 2\pi$, and (iv) find the area enclosed by the curve.

- We easily compute $x'(t) = -2\sin(t)$ and $y'(t) = 3\cos(t)$.
- By the first formula, the slope of the tangent line to the circle at $(\cos(t), \sin(t))$ is $-\dfrac{3\cos(t)}{2\sin(t)}$.
  - When $t = \pi/6$, the slope of the tangent line is $-\dfrac{3}{2}$. The tangent line also passes through the point $(2\cos(t),\, 3\sin(t)) = \left(\sqrt{2},\, \dfrac{3\sqrt{2}}{2}\right)$, so it has equation $\boxed{y - \dfrac{3\sqrt{2}}{2} = -\dfrac{3}{2} \cdot \left(x - \sqrt{2}\right)}$.
  - Like with the circle, the expression for the slope is zero when $\cos(t) = 0$, which occurs for $t = \pi/2$, $3\pi/2$, $5\pi/2$, ... This gives rise to two points on the curve where there is a horizontal tangent line: $\boxed{(0,3) \text{ and } (0,-3)}$.
  - Similarly, the slope will be undefined when $\sin(t) = 0$, which occurs for $t = 0$, π, 2π, ..., which also gives rise to two points on the curve with a vertical tangent line: $\boxed{(2,0) \text{ and } (2,0)}$.
- By the second formula, the arclength of the ellipse is

$$\int_0^{2\pi} \sqrt{[-2\sin(t)]^2 + [3\cos(t)]^2}\, dt = \boxed{\int_0^{2\pi} \sqrt{4\sin^2 t + 9\cos^2 t}\, dt}.$$

  This integral cannot be evaluated using elementary techniques. Its approximate value is 15.8654.
- By the third formula, the area of the ellipse is

$$-\int_0^{2\pi} 3\sin(t) \cdot (-2\sin(t))\, dt = \int_0^{2\pi} 6\sin^2(t)\, dt = 6\int_0^{2\pi} \frac{1-\cos(2t)}{2}\, dt = \boxed{6\pi}.$$

- Example: For the hypocycloid $x = 8\cos(t) + \cos(8t)$, $y = 8\sin(t) - \sin(8t)$, find (i) the arclength of the curve, and (ii) the area enclosed by the curve.
  - We compute $x'(t) = -8\sin(t) - 8\sin(8t)$ and $y'(t) = 8\cos(t) - 8\cos(8t)$.
  - After using some trigonometric identities, we can compute that the arclength of the curve is

$$\begin{aligned}
\int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt &= \int_0^{2\pi} \sqrt{[-8\sin(t) - 8\sin(8t)]^2 + [8\cos(t) - 8\cos(8t)]^2}\, dt \\
&= 8\int_0^{2\pi} \sqrt{2 + 2\sin(t)\sin(8t) - 2\cos(t)\cos(8t)}\, dt \\
&= 8\int_0^{2\pi} \sqrt{2 - 2\cos(9t)}\, dt \\
&= 8\int_0^{2\pi} 2\left|\sin\left(\frac{9t}{2}\right)\right|\, dt \\
&= 16 \cdot \left[9\int_0^{2\pi/9} \sin\left(\frac{9t}{2}\right)\, dt\right] \\
&= 16 \cdot 9 \cdot \frac{4}{9} \\
&= \boxed{64}.
\end{aligned}$$

  This is interesting, since (perhaps) we might not have expected this value to be at all nice, let alone an integer. (Especially given that the curve itself was obtained by a circle rolling around another circle.)

∘ The area inside the curve is

$$\begin{aligned}
-\int_0^{2\pi} y(t)\cdot x'(t)\,dt &= -\int_0^{2\pi} [8\sin(t)-\sin(8t)]\cdot[-8\sin(t)-8\sin(8t)]\,dt \\
&= 8\int_0^{2\pi}\left[8\sin^2(t)+7\sin(t)\sin(8t)-\sin^2(8t)\right]\,dt \\
&= 8\int_0^{2\pi}\left[8\cdot\frac{1-\cos(2t)}{2}+7\cdot\frac{\cos(7t)-\cos(9t)}{2}-\frac{1-\cos(16t)}{2}\right]\,dt \\
&= 8\cdot\frac{7}{2}\cdot 2\pi \\
&= \boxed{56\pi}.
\end{aligned}$$

This is interesting, since it's also a pretty nice number. Also, we see that this curve has an integral arclength but encloses an area that's a multiple of π.

6.3 Polar Coordinates

- Polar coordinates are an alternative coordinate system to the typical rectangular, Cartesian, xy-coordinate system.
- Points in polar form are expressed in terms of their distance r to the "pole" (i.e., the origin) and the angle θ they make with respect to a fixed line (the positive x-axis) through the pole.
  - ∘ The polar point (r,θ) converts to the rectangular point (x,y) with $\boxed{x = r\cdot\cos(\theta)}$ and $\boxed{y = r\cdot\sin(\theta)}$.
  - ∘ The Cartesian point (x,y) converts to the polar point (r,θ) with $\boxed{r=\sqrt{x^2+y^2}}$ and $\boxed{\theta=\tan^{-1}\left(\frac{y}{x}\right)}$ if $x \geq 0$ and $\boxed{\theta=\tan^{-1}\left(\frac{y}{x}\right)+\pi}$ if $x < 0$.
  - ∘ Note/Warning: In this class, we allow negative values of r, with the convention that $(-r,\theta)=(r,\pi-\theta)$ are the same (polar) point. This assumption is nonstandard, and other uses of polar coordinates may not allow for negative values of r.
- Here are a few examples.
  - ∘ Example: The rectangular point $(1,1)$ is the point $(\sqrt{2},\frac{\pi}{4})$ in polar.
  - ∘ Example: The rectangular point $(0,-11)$ is the point $(11,\frac{3\pi}{2})$ in polar.
  - ∘ Example: The polar point $(4,\frac{5\pi}{6})$ is the point $(-2\sqrt{3},2)$ in rectangular.
  - ∘ Example: The polar point $(12,\frac{11\pi}{4})$ is the point $(-6\sqrt{2},6\sqrt{2})$ in rectangular.
- Note that, unlike in rectangular coordinates, points have several "different" coordinates in polar form.
  - ∘ For example, $(r,\theta)=(r,\theta+2\pi)$ for any r and θ. Since shifting θ by 2π, or by any multiple of 2π, yields the same point in the plane, we will usually assume that θ is in the interval $[0,2\pi]$.
  - ∘ Also, notice that the point $(0,\theta)$ for any value of θ is the origin.
- Polar coordinates are used frequently in applications of calculus, especially in physics and engineering.
  - ∘ Polar coordinates are very useful for describing certain kinds of curves (like the conic sections: ellipses, parabolas, and hyperbolas) more simply than in rectangular coordinates.
  - ∘ Since planetary orbits are described by conic sections, many computations in planetary astrophysics can be greatly simplified when expressed in polar coordinates.

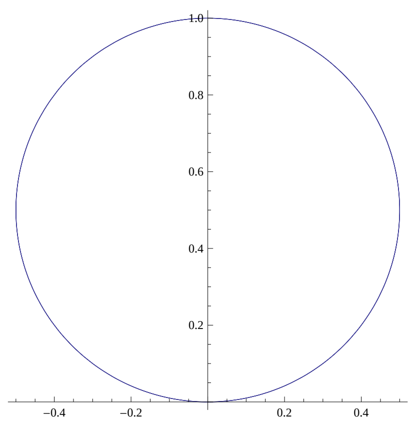
- Likewise, in any setting with some kind of radial motion (e.g., the motion of an object revolving around a circle, or the motion of a joint around a fixed pivot), polar coordinates are often more natural than rectangular coordinates.
- In multivariable calculus, polar coordinates (and generalizations) are often used as a "change of coordinates" to simplify certain kinds of integrals.

- Just like with rectangular coordinate systems, we can describe curves with an equation involving the two coordinates.
  - Frequently, we give equations in the form $r = f(\theta)$ for some function $f(\theta)$.
  - This kind of polar curve can also be thought of as a parametric curve, by writing $x = r\cos(\theta) = f(\theta)\cos(\theta)$ and $y = r\sin(\theta) = f(\theta)\sin(\theta)$, where θ is now the parameter (in place of t).
  - This will be extremely useful very soon when we want to compute arclengths, slopes of tangents, and areas.

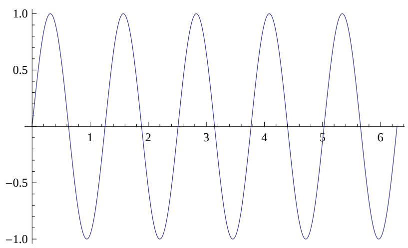
- To graph a polar curve, the simplest thing to do is just plug in values of θ to compile a list of points (r, θ) and then join them up with a smooth curve.

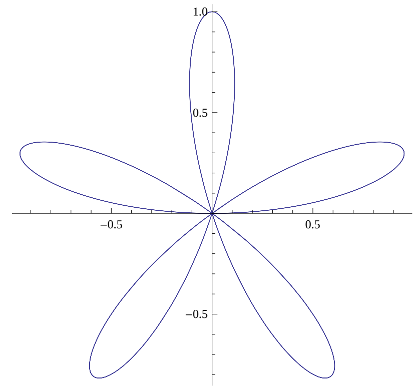
- A slightly better way is to graph r as a function of θ (in the traditional "Cartesian" sense), and then to use the features of this graph to draw the polar curve $r = f(\theta)$.

- Example: The curve $r = \sin(\theta)$ is a circle of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$.
  - The fact that the curve is a circle can be seen by converting to rectangular coordinates: the graph is the same as that of $r^2 = r\sin(\theta)$, which is just $x^2 + y^2 = y$.

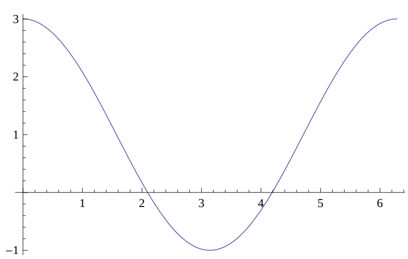


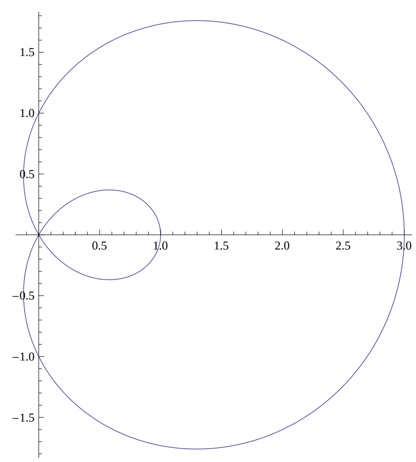
- Example: The curve $r = \sin(5\theta)$ is a five-petaled rose. The graph of r as a function of θ is given below, as is the actual polar graph.
  - To produce the polar graph by hand, use the features of the $r\theta$-graph: we can see that as θ increases from 0 through $\pi/10$ to $\pi/5$ the curve swings out to a radius of 1 and then back to the origin: this gives a 'petal' in the first quadrant. Tracing the curve further from $\pi/5$ to $2\pi/5$ gives another petal in the third quadrant. We continue through three more petals, until we get all the way to $\theta = 2\pi$, when we would start retracing the beginning of the curve.



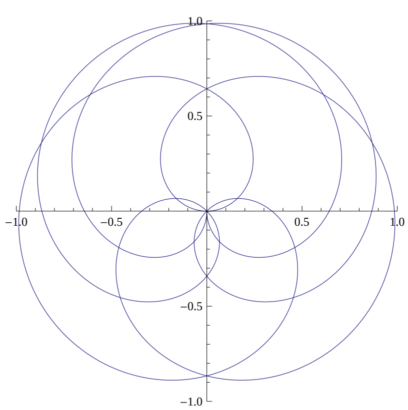


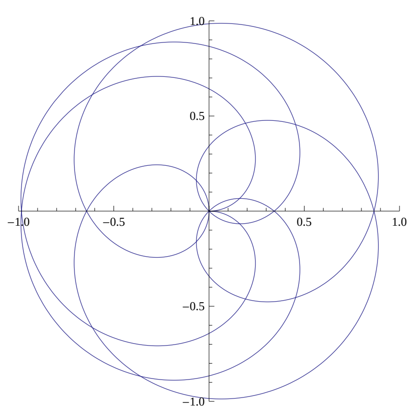
- Example: The curve $r = 1 + 2\cos(\theta)$ is called a limaçon (Old French for "snail", which is vaguely what the graph looks like). The $r\theta$-graph and the polar graph are given below.
  - To produce the polar graph, we start following the $r\theta$-graph at $\theta = 0$, which has $r = 3$. The polar curve traces around a counterclockwise spiral, approaching the origin and passing through it when the $r\theta$-graph crosses zero, which occurs at $\theta = 2\pi/3$. The polar curve then traces clockwise, passing through the origin again at $\theta = 4\pi/3$, and then spirals out and back toward the starting point with $\theta = 2\pi$ and $r = 3$.

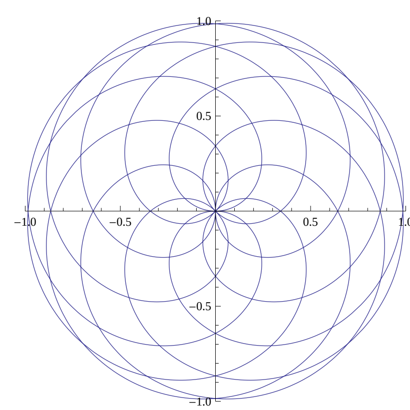




- Example: The curve $r = \sin(4\theta/9)$ for $0 \le \theta \le 18\pi$ produces some rather pretty and symmetric graphs. The graphs on the intervals $[-\frac{9}{2}\pi, \frac{9}{2}\pi]$, $[0, 9\pi]$, and $[0, 18\pi]$ are given below.







- We can also convert Cartesian equations into polar ones, simply by setting $x = r\cos\theta$ and $y = r\sin\theta$.
- Example: Give a polar equation for the line $y = x$.

- Substituting for the polar variables yields $r\cos\theta = r\sin\theta$.
- This is a perfectly good equation already, but we can simplify it: clearing the r yields $\cos\theta = \sin\theta$, and this is equivalent to $\tan\theta = 1$.
- Taking the arctangent of both sides gives $\boxed{\theta = \frac{\pi}{4}, \frac{5\pi}{4}}$. Notice that the equation $\theta = \frac{\pi}{4}$ gives the portion of the line in the first quadrant, while the equation $\theta = \frac{5\pi}{4}$ gives the portion of the line in the third quadrant: we need both in order to get the entire line.
- More generally, the Cartesian equation $y = ax$ is equivalent to the polar equation $\tan\theta = 1/a$.

- Example: Give a polar equation for the implicit curve $(x^2 + y^2)^2 = xy$.
  - Notice that $x^2 + y^2 = r^2$, and $x = r\cos(\theta)$ with $y = r\sin(\theta)$.
  - Plugging these in yields $(r^2)^2 = r\cos(\theta) \cdot r\sin(\theta)$.
  - Cancelling r^2 yields $\boxed{r^2 = \cos(\theta)\sin(\theta)}$. This can equivalently be written as $\boxed{r^2 = \frac{1}{2}\sin(2\theta)}$.

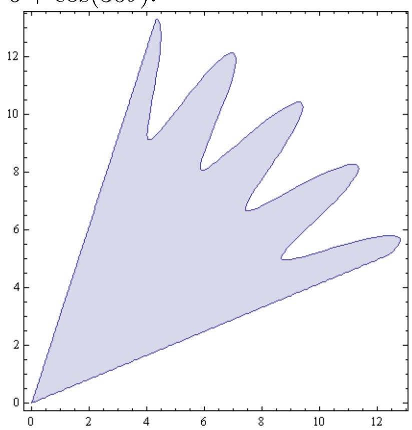
6.4 Tangent Lines, Arclengths, and Areas for Polar Curves

- As with parametric curves, once we know what a polar curve looks like, we can ask our three basic calculus questions:
  1. How can we find the tangent line to the curve?
  2. How can we find the arclength of the curve?
  3. How can we find the area underneath (or inside, around, described by...) the curve?
- As before, we can answer each of these questions. In fact, all of the answers are simply applications of the formulas we found earlier for parametric curves:
- Theorem: The slope $\frac{dy}{dx}$ of the tangent line to the curve $r = f(\theta)$ is given by $\boxed{\frac{dy}{dx} = \frac{r' \cdot \sin(\theta) + r \cdot \cos(\theta)}{r' \cdot \cos(\theta) - r \cdot \sin(\theta)}}$, where r' means $f'(\theta)$.
  - Proof: We have $x = r \cdot \cos(\theta)$ and $y = r \cdot \sin(\theta)$. Thus we have $dx/d\theta = r' \cdot \cos(\theta) - r \cdot \sin(\theta)$ and $dy/d\theta = r' \cdot \sin(\theta) + r \cdot \cos(\theta)$. Plugging into the parametric tangent slope formula gives the answer above.
  - Note: Although it doesn't often come up, this formula also works even if r is not given as an explicit function of θ. If we instead had some kind of implicit equation like $r^2 + \theta^2 = 1$, we could use implicit differentiation to find r', and use that in the formula.
- Theorem: The arclength s of the curve $r = f(\theta)$ for $\theta_1 \le \theta \le \theta_2$ is given by the integral $\boxed{s = \int_{\theta_1}^{\theta_2} \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\, d\theta}$.
  - Important Warning: When simplifying the square root, do not forget that $\sqrt{x^2} = |x|$, not x. An arclength can never be a negative number!
  - Proof: We have $dx/d\theta = r' \cdot \cos(\theta) - r \cdot \sin(\theta)$ and $dy/d\theta = r' \cdot \sin(\theta) + r \cdot \cos(\theta)$. Then

$$\begin{aligned}
[x'(\theta)]^2 + [y'(\theta)]^2 &= \left[(r')^2\cos^2\theta - 2rr'\sin\theta\cos\theta + r^2\sin^2\theta\right] + \left[r^2\cos^2\theta + 2rr'\sin\theta\cos\theta + (r')^2\sin^2\theta\right] \\
&= (r')^2\left[\cos^2\theta + \sin^2\theta\right] + r^2\left[\sin^2\theta + \cos^2\theta\right] \\
&= (r')^2 + r^2
\end{aligned}$$

and plugging this into the parametric arclength formula gives the result.

- Theorem: The area A inside the closed curve $r = f(\theta)$ for $t_1 \le \theta \le t_2$ is given by the integral $\boxed{A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta}$, provided the curve is traveling counterclockwise as θ increases, and does not cross itself or reverse direction.
  - Important Warning: As with parametric curves, be exceedingly careful if the curve changes direction as it is being traced out, or if the curve crosses over itself. (Otherwise, the area may be off by a factor of -1.) It may be necessary to break up the region of integration to ensure the correct answer is found.
  - Note: If the curve is not closed, the integral will give the area of the "radial sector" defined by the inequalities $0 \le r \le f(\theta)$ and $\theta_1 \le \theta \le \theta_2$. Here is an illustration of this region for the curve $r = 6 + \cos(30\theta)$:



  - Proof: One can prove this theorem by converting the "area differential" $dA = dx\, dy$ into polar coordinates; this will give $dA = r\, dr\, d\theta$, which upon integrating turns into $\frac{1}{2}r^2 d\theta$. There is also a slicker proof using the "alternate formula" for area in parametric coordinates: $A = \int_{t_1}^{t_2} \frac{1}{2}\left[x(t)\cdot y'(t) - x'(t)\cdot y(t)\right]\, dt$. Replacing t with θ and using the expressions for $x(\theta)$, $x'(\theta)$, $y(\theta)$, $y'(\theta)$ yields

$$\begin{aligned} x(t)\cdot y'(t) - x'(t)\cdot y(t) &= [r\cos(\theta)]\cdot[r'\cdot\sin(\theta) + r\cdot\cos(\theta)] - [r'\cdot\cos(\theta) - r\cdot\sin(\theta)]\cdot[r\sin(\theta)] \\ &= r^2\cdot\left[\cos^2(\theta) + \sin^2(\theta)\right] \\ &= r^2 \end{aligned}$$

    which (after plugging in to the integral) gives the polar area formula.

- Example: For the five-petaled rose $r = \sin(5\theta)$, find (i) a Cartesian equation for the tangent line to the curve at $\theta = \pi/4$, (ii) an expression for the arclength of one petal, and (iii) the area enclosed by one petal.
  - By the first formula, the slope of the tangent line to the curve is given by $\frac{dy}{dx} = \frac{r'\cdot\sin(\theta) + r\cdot\cos(\theta)}{r'\cdot\cos(\theta) - r\cdot\sin(\theta)} = \frac{5\cos(5\theta)\sin(\theta) + \sin(5\theta)\cos(\theta)}{5\cos(5\theta)\cos(\theta) - \sin(5\theta)\sin(\theta)}$. (This doesn't simplify to anything nice.)
    - Thus, when $\theta = \pi/4$, we see that the tangent line has slope $\frac{5\left(-\sqrt{2}/2\right)\left(\sqrt{2}/2\right) + (-\sqrt{2}/2)(\sqrt{2}/2)}{5(-\sqrt{2}/2)(\sqrt{2}/2) - (-\sqrt{2}/2)(\sqrt{2}/2)} = \frac{3}{2}$.
    - The tangent line also passes through the polar point $(r,\theta) = \left(-\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$, which is the point $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ in rectangular coordinates.
    - Hence the tangent line has equation $\boxed{y + \frac{1}{2} = \frac{3}{2}\left(x + \frac{1}{2}\right)}$ in rectangular coordinates.

- As we saw when we graphed it earlier, the range $0 \le \theta \le \frac{2\pi}{5}$ sweeps out one petal of the rose.
- Thus, by the given formulas, the arclength of one petal is given by the integral $\boxed{\int_0^{2\pi/5} \sqrt{\sin^2(5\theta) + 25\cos^2(\theta)} d\theta}$.
  - This integral does not simplify and cannot be evaluated using elementary methods. The approximate value of the arclength is 4.2020.
- The area of one petal of the rose is given by the integral $\frac{1}{2}\int_0^{2\pi/5} \sin^2(5\theta)\, d\theta = \frac{1}{2}\int_0^{2\pi/5} \frac{1-\cos(10\theta)}{2}\, d\theta = \boxed{\frac{\pi}{10}}$.

• Example: For the limaçon $r = 1 + 2\cos(\theta)$, find (i) a Cartesian equation for the tangent line to the curve at $\theta = \pi/2$, (ii) the values of r where the tangent line is vertical, (iii) an integral for the arclength of the entire curve, (iv) the area enclosed by the inner loop, and (v) the area enclosed by the entire curve, including the inner loop.

- Note that the full curve is given by $0 \le \theta \le 2\pi$, and the inner loop is bounded by $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$.
- The slope of the tangent line to the curve is given by $\frac{dy}{dx} = \frac{r' \cdot \sin(\theta) + r \cdot \cos(\theta)}{r' \cdot \cos(\theta) - r \cdot \sin(\theta)} = -\frac{\cos(\theta) + 2\cos(2\theta)}{\sin(\theta) + 2\sin(2\theta)}$.
  - When $\theta = \pi/2$, the slope is equal to $-\frac{0-2}{1+0} = 2$. The tangent line also passes through the polar point $(r, \theta) = \left(1, \frac{\pi}{2}\right)$, which is the point $(0, 1)$ in rectangular coordinates. Hence by point-slope, the tangent line has equation $\boxed{y - 1 = 2x}$.
  - Furthermore, the denominator of the slope expression is equal to $\sin(\theta) + 4\sin(\theta)\cos(\theta) = \sin(\theta) \cdot (1 + 4\cos(\theta))$, which is zero when $\sin(\theta) = 0$ or when $\cos(\theta) = -\frac{1}{4}$. In the first case, $r = \boxed{\pm 1}$, and in the second case, $r = \boxed{\frac{1}{2}}$.
- The arclength of the whole curve is given by $\boxed{\int_0^{2\pi} \sqrt{(1 + 2\cos(\theta))^2 + (-2\sin(\theta))^2}\, d\theta}$.
  - The approximate value of the whole curve's arclength is $\boxed{13.3649}$, while the portion bounding the inner loop has length (approximately) $\boxed{2.6825}$.
- We first compute the area of the "inner loop" of the limaçon by evaluating

$$\begin{aligned} \frac{1}{2}\int_{2\pi/3}^{4\pi/3} [1 + 2\cos(\theta)]^2\, d\theta &= \frac{1}{2}\int_{2\pi/3}^{4\pi/3} [1 + 4\cos(\theta) + 2(1 + \cos(2\theta))]\, d\theta \\ &= \frac{1}{2}\left[3\theta + 4\sin(\theta) + \sin(2\theta)\right]\Big|_{\theta=2\pi/3}^{4\pi/3} = \boxed{\pi - \frac{3\sqrt{3}}{2}}. \end{aligned}$$

- For the last part, it would first appear that the area enclosed by the entire curve is given by

$$\begin{aligned} \frac{1}{2}\int_0^{2\pi} [1 + 2\cos(\theta)]^2\, d\theta &= \frac{1}{2}\int_0^{2\pi} [1 + 4\cos(\theta) + 2(1 + \cos(2\theta))]\, d\theta \\ &= \frac{1}{2}\left[3\theta + 4\sin(\theta) + \sin(2\theta)\right]\Big|_{\theta=0}^{2\pi} = 3\pi. \end{aligned}$$

- However, it is critical to note that, due to the way the curve is traced out, this area excludes the "inner loop". We must add back in this missing portion, to obtain the final result of $\boxed{4\pi - \frac{3\sqrt{3}}{2}}$ for the total area.

6.5 Arithmetic with Complex Numbers

- Complex numbers may seem daunting, arbitrary, and strange when first introduced, but they are (in fact) very useful in mathematics and elsewhere. Plus, they're just neat.
- Some History: Complex numbers were first encountered by mathematicians in the 1500s who were trying to write down general formulas for solving cubic equations (i.e., equations like $x^3 + x + 1 = 0$), in analogy with the well-known formula for the solutions of a quadratic equation. It turned out that their formulas required manipulation of complex numbers, even when the cubics they were solving had three real roots.
  - It took over 100 years before complex numbers were accepted as something mathematically legitimate: even negative numbers were sometimes suspect, so (as the reader may imagine) their square roots were even more questionable.
  - The stigma is still evident even today in the terminology ("imaginary numbers"), and the fact that complex numbers are often glossed over or ignored in mathematics courses.
  - Nonetheless, they are very real objects (no pun intended), and have a wide range of uses in mathematics, physics, and engineering.
  - Among neat applications of complex numbers are deriving trigonometric identities with much less work (see later) and evaluating certain kinds of definite and indefinite integrals. For example, using the theory of functions of a complex variable, one can derive many rather unusual results, such as $\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\, dx = \frac{\pi}{e}$.
- Definitions: A complex number is a number of the form $a + bi$, where a and b are real numbers and i is the "imaginary unit", defined so that $i^2 = -1$.
  - Notation: Sometimes, i is written as $\sqrt{-1}$. In certain areas (especially electrical engineering), the letter j can be used to denote $\sqrt{-1}$, rather than i (which is used to denote electrical current).
  - The real part of $z = a + bi$, denoted $\mathrm{Re}(z)$, is the real number a.
  - The imaginary part of $z = a + bi$, denoted $\mathrm{Im}(z)$, is the real number b.
  - The complex conjugate of $z = a + bi$, denoted $\overline{z}$, is the complex number $a - bi$.
    - The notation for conjugate varies among disciplines. The notation z^* is often used in physics and computer programming to denote the complex conjugate, in place of $\overline{z}$.
  - The modulus (also called the absolute value, magnitude, or length) of $z = a + bi$, denoted $|z|$, is the real number $\sqrt{a^2 + b^2}$.
  - Example: $\mathrm{Re}(4 - 3i) = 4$, $\mathrm{Im}(4 - 3i) = -3$, $\overline{4 - 3i} = 4 + 3i$, $|4 - 3i| = 5$.
- Two complex numbers are added (or subtracted) simply by adding (or subtracting) their real and imaginary parts: $(a + bi) + (c + di) = (a + c) + (b + d)i$.
  - Example: The sum of $1 + 2i$ and $3 - 4i$ is $\boxed{4 - 2i}$. The difference is $(1 + 2i) - (3 - 4i) = \boxed{-2 + 6i}$.
- Two complex numbers are multiplied using the distributive law and the fact that $i^2 = -1$: $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$.
  - Example: The product of $1 + 2i$ and $3 - 4i$ is $(1 + 2i)(3 - 4i) = 3 + 6i - 4i - 8i^2 = \boxed{11 + 2i}$.
- For division, we rationalize the denominator: $\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i$.
  - Example: The quotient of $2i$ by $1 - i$ is $\frac{2i}{1 - i} = \frac{2i(1 + i)}{(1 - i)(1 + i)} = \frac{-2 + 2i}{2} = \boxed{-1 + i}$.
- A key property of the conjugate is that it is multiplicative: if $z = a + bi$ and $w = c + di$, then $\overline{zw} = \overline{z} \cdot \overline{w}$. (This is easy to see just by multiplying out the relevant quantities.) From this we see that the modulus is also multiplicative: $|zw| = |z| \cdot |w|$.

- Example: If $z = 1 + 2i$ and $w = 3 - i$, then $\overline{z} = 1 - 2i$ and $\overline{w} = 3 + i$. We compute $zw = 5 + 5i$ and $\overline{z} \cdot \overline{w} = 5 - 5i$, so indeed $\overline{zw} = \overline{z} \cdot \overline{w}$. Furthermore, we have $|z| = \sqrt{5}, |w| = \sqrt{10}$, and $|zw| = \sqrt{50} = |z| \cdot |w|$.
- This is the underlying reason for why division works in general: we write $\frac{z}{w} = \frac{z \cdot \overline{w}}{w \cdot \overline{w}} = \frac{z \cdot \overline{w}}{|w|^2}$, where the denominator is now the real number $|w|^2 = c^2 + d^2$.

- Using complex numbers, we can give meaning to the solutions of a quadratic equation even when they are not real numbers.
  - Explicitly, by completing the square, we see that the polynomial $az^2 + bz + c = 0$ has the two solutions $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ over the complex numbers. (Technically, this is not quite true when $b^2 - 4ac = 0$: in this case, the convention is to say that this polynomial still has two roots, but they are equal.)
  - Notice that the expression for the roots now makes sense even if $b^2 - 4ac < 0$: the roots are simply non-real complex numbers.
  - We can then factor the polynomial as $az^2 + bz + c = a(z - r_1)(z - r_2)$ where $r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ are the two roots.
  - More generally, the Fundamental Theorem of Algebra says that any polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ can be completely factored into a product of linear terms over the complex numbers. (This a foundational result in algebra and it was first correctly proven by Argand and Gauss in the early 1800s.)

6.6 Complex Exponentials, Polar Form, and Euler's Theorem

- We often think of the real numbers geometrically, as a line. The natural way to think of the complex numbers is as a plane, with the x-coordinate denoting the real part and the y-coordinate denoting the imaginary part.
  - Once we do this, there is a natural connection to polar coordinates: namely, if $z = x + yi$ is a complex number which we identify with the point (x, y) in the complex plane, then the modulus $|z| = \sqrt{x^2 + y^2}$ is simply the coordinate r when we convert (x, y) from Cartesian to polar coordinates.
  - Furthermore, if we are given that $|z| = r$, we can uniquely identify z given the angle θ that the line connecting z to the origin makes with the positive real axis. (This is the same θ from polar coordinates.)

- From our computations with polar coordinates (or simple trigonometry), we see that we can write z in the form $\boxed{z = r \cdot [\cos(\theta) + i \cdot \sin(\theta)]}$.
  - This is called the polar form of z. The angle θ is called the argument of z and sometimes denoted $\theta = \arg(z)$.
  - Notational remark: Since it comes up frequently, some people like to abbreviate $\cos(\theta) + i \cdot \sin(\theta)$ by $\operatorname{cis}(\theta)$ ("cosine-i-sine").
  - Conversely, if we know $z = x + iy$ then we can compute the (r, θ) form fairly easily, since $r = |z|$ and $\theta = \arg(z)$.
  - Explicitly, we have $\boxed{r = \sqrt{x^2 + y^2}}$ and $\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$ if $x > 0$, and $\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi}$ if $x < 0$.
    - This extra $+\pi$ is needed because of the specific way we've chosen the definition of arctangent. Otherwise we'd get the wrong value for θ if z lies in the second or third quadrants.
    - Again, note that these are the exact same formulas for converting between rectangular and polar coordinates.
  - Example: If $z = 1 + i$, then the corresponding values of r and θ are $r = |z| = \sqrt{2}$ and $\theta = \tan^{-1}(1) = \frac{\pi}{4}$, so we can write z in polar form as $z = \boxed{\sqrt{2} \cdot \left[\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right]}$. (It is easy to multiply this out and verify that the result is simply $1 + i$.)

- We're comfortable with plugging complex numbers into polynomials, but what about other functions? We'd like to be able to say what something like e^{a+bi} should mean.
  - We feel like e^{a+bi} should obey the exponential rules, and so we want to say $e^{a+bi} = e^a \cdot e^{bi}$. So really, we only care about what e^{bi} is.
- The key result is what is called <u>Euler's identity</u>: $\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$.
  - Notice that this is the same expression that showed up in the polar form of a complex number.
- Euler's identity encodes <u>a lot</u> of information. Here is one application:
  - Exponential rules state $e^{i(\theta+\varphi)} = e^{i\theta} \cdot e^{i\varphi}$.
  - Expanding out both sides with Euler's identity yields
  $$\cos(\theta+\varphi) + i \cdot \sin(\theta+\varphi) = [\cos(\theta) + i \cdot \sin(\theta)] \cdot [\cos(\varphi) + i \cdot \sin(\varphi)].$$
  - Multiplying out and simplifying yields
  $$\cos(\theta+\varphi) + i \cdot \sin(\theta+\varphi) = [\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)] + i \cdot [\cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi)].$$
  - Setting the real and imaginary parts equal yields (respectively) the equalities
  $$\begin{aligned} \cos(\theta+\varphi) &= \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) \\ \sin(\theta+\varphi) &= \cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi) \end{aligned}$$
  and notice that these are exactly the addition formulas for sine and cosine!
  - What this means is that the rather strange-looking trigonometric addition formulas, which are rather weird and arbitrary when first encountered, actually just reflect the natural structure of the multiplication of complex numbers.
- Another application is the simple relation $e^{i(n\theta)} = (e^{i\theta})^n$. Writing out both sides in terms of sines and cosines gives <u>De Moivre's identity</u> $\boxed{\cos(n\theta) + i \cdot \sin(n\theta) = [\cos(\theta) + i \cdot \sin(\theta)]^n}$.
  - Plugging in various values of n and then expanding out the right-hand side via the Binomial Theorem allows one to obtain identities for $\sin(n\theta)$ and $\cos(n\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$.
  - <u>Example</u>: $\cos(2\theta) + i \cdot \sin(2\theta) = [\cos(\theta) + i \cdot \sin(\theta)]^2 = \left(\cos^2\theta - \sin^2\theta\right) + i \cdot (2\sin\theta\cos\theta)$, and so we recover the double-angle formulas for sine and cosine.
- Even setting $\theta = \pi$ in Euler's identity tells us something very interesting: we obtain $e^{i\pi} = -1$, or, better, $\boxed{e^{i\pi} + 1 = 0}$.
  - The constants 0, 1, i, e, and π are, without a doubt, the five most important numbers in all of mathematics.
  - That there exists one simple equation relating all five of them is (to the author at least) quite amazing.
- Using Euler's identity and the polar form of complex numbers above, we see that every complex number can be written as $\boxed{z = r \cdot e^{i\theta}}$ for some r and θ. We call this the <u>exponential form</u> of z.
  - <u>Example</u>: We can draw $1+i$ in the complex plane, or use the formulas, to see that $|1+i| = \sqrt{2}$ and $\arg(1+i) = \frac{\pi}{4}$, and so we see that $\boxed{1+i = \sqrt{2} \cdot e^{i\pi/4}}$.
  - <u>Example</u>: Either by geometry or trigonometry, we see that $\left|1 - i\sqrt{3}\right| = 2$ and $\arg(1-i\sqrt{3}) = -\frac{\pi}{3}$, hence $\boxed{1+i\sqrt{3} = 2 \cdot e^{-i\pi/3}}$.
- It is very easy to take powers of complex numbers when they are in exponential form: $(r \cdot e^{i\theta})^n = r^n \cdot e^{i(n\theta)}$.

- Example: Compute $(1+i)^8$.
    - From above we have $1+i=\sqrt{2}\cdot e^{i\pi/4}$, so $(1+i)^8=\left(\sqrt{2}\cdot e^{i\pi/4}\right)^8=(\sqrt{2})^8\cdot e^{8i\pi/4}=2^4\cdot e^{2i\pi}=\boxed{16}$.
    - Note how much easier this is compared to multiplying $(1+i)$ by itself eight times.
- Example: Compute $(1-i\sqrt{3})^9$.
    - From above we have $1-i\sqrt{3}=2\cdot e^{-i\pi/3}$, so $(1-i\sqrt{3})^9=2^9\cdot e^{-9i\pi/3}=512\cdot e^{-3i\pi}=\boxed{-512}$.

- Taking roots of complex numbers is also easy using the polar form. We do need to be slightly careful, since (like having 2 possible square roots of a positive real number), there are n different nth roots of any nonzero complex number.
    - The general formula says that the n possible nth roots of $z=r\cdot e^{i\theta}$ are $\boxed{\sqrt[n]{r}e^{i\theta/n}\cdot e^{2i\pi k/n}}$, where k ranges through $0,1,\cdots,n-1$.
    - One can check that the nth power of all of these numbers is indeed $r\cdot e^{i\theta}$, since $e^{2i\pi k}=1$ (for k an integer) by Euler's formula. And they are clearly all distinct, and so they are all of the nth roots.

- Example: Find all complex square roots of $2i$.
    - We are looking for square roots of $2i=2\cdot e^{i\pi/2}$. By the formula, the two square roots are $\sqrt{2}\cdot e^{i[\pi/4+k\pi]}$ for $k=0,1$.
    - Converting from exponential to rectangular form using Euler's formula gives the two square roots as $\boxed{1+i,-1-i}$.
    - Indeed, we can easily multiply out to verify that $(1+i)^2=(-1-i)^2=2i$, as it should be.

- Example: Find all complex numbers $z=a+bi$ with $z^3=1$.
    - We are looking for cube roots of $1=1\cdot e^0$. By the formula, the three cube roots of 1 are $1\cdot e^{2ki\pi/3}$, for $k=0,1,2$.
    - Converting from exponential to rectangular form using Euler's formula gives the roots as $\boxed{1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i}$ in $x+yi$ form.

Well, you're at the end of my handout. Hope it was helpful.