Calculus I (part 3): Applications of Differentiation (by Evan Dummit, 2019, v. 2.50)

Contents

3 Applications of Differentiation

Derivatives have a wide variety of applications. We will begin by discussing two closely related, and fundamental, uses of the first derivative: that of finding the largest and smallest values attained by a differentiable function, and that of understanding where a function is increasing or decreasing. Along the way, we will prove a number of fundamental results about derivatives, including Rolle's Theorem, the Mean Value Theorem, and the First Derivative Test. We will then turn our attention to the second derivative and use it to study concavity, and discuss how to draw accurate graphs of functions.

Afterward, we discuss some other applications: L'Hôpital's rule for evaluating indeterminate limits, applied optimization, and antiderivatives with applications to the sciences.

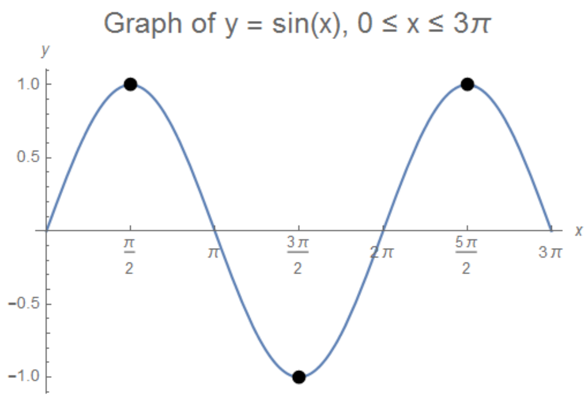
3.1 Minimum and Maximum Values

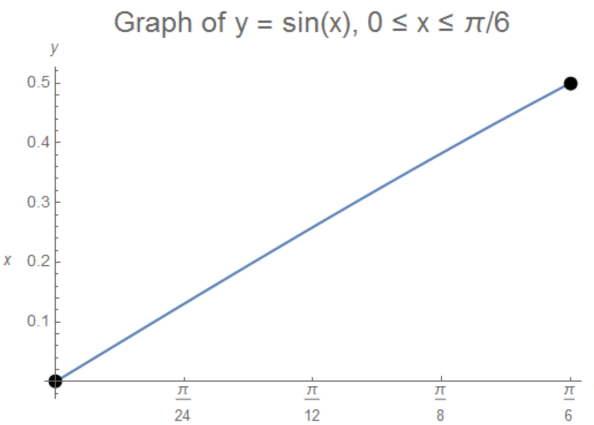
- In this section, our goal is to describe how to use calculus to find minimum and maximum values of functions.

3.1.1 Absolute Minimum and Maximum Values

- We are interested in finding minimum and maximum values, so we must first define what precisely this means:
- Definitions: Suppose that $f(x)$ is a function defined on an interval I. We say f has an absolute maximum on I at $x = d$ if $f(x) \le f(d)$ for all x in I. We say f has an absolute minimum on I at $x = c$ if $f(x) \ge f(c)$ for all x in I.

- Terminology: The plural of "maximum" is "maxima" and the plural of "minimum" is "minima". The word "extremum" is used to refer to a point that is either a minimum or a maximum (also called "extreme points"). The plural of "extremum" is "extrema".
- We note that an absolute maximum need not be unique: if f takes the same maximum value for several values of x in I, all of them are considered absolute maxima of f. (The same holds for minimum values.)
- We also note that the absolute minimum and maximum of a particular function on an interval will depend on the interval being considered.
- Example: On the interval $[0, 3\pi]$, the function $f(x) = \sin(x)$ has an absolute maximum value of 1, occurring both at $x = \pi/2$ and $x = 5\pi/2$, and has an absolute minimum value of -1 occurring at $x = 3\pi/2$.

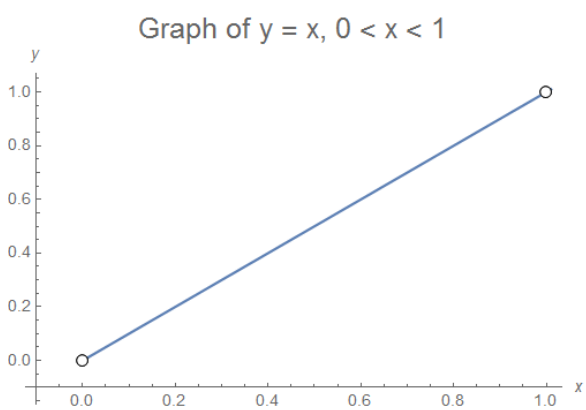


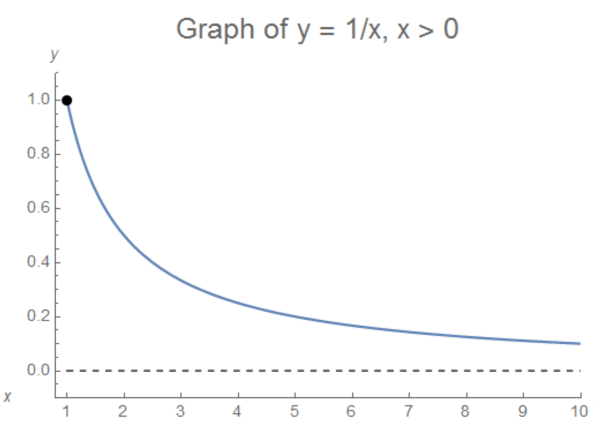


- Example: On the interval $[0, \pi/6]$, the function $f(x) = \sin(x)$ has an absolute maximum value of $1/2$, occurring at $x = \pi/6$, and has an absolute minimum value of 0 occurring at $x = 0$.

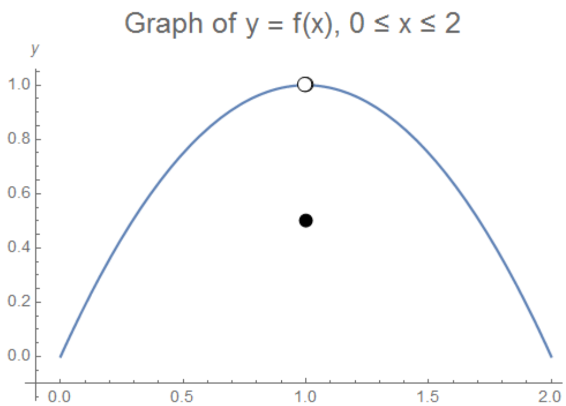
• It may seem obvious that, given an interval I and a function f, that f will possess an absolute minimum and maximum somewhere in I. However, this is not true! Here are a few examples illustrating what can go wrong:

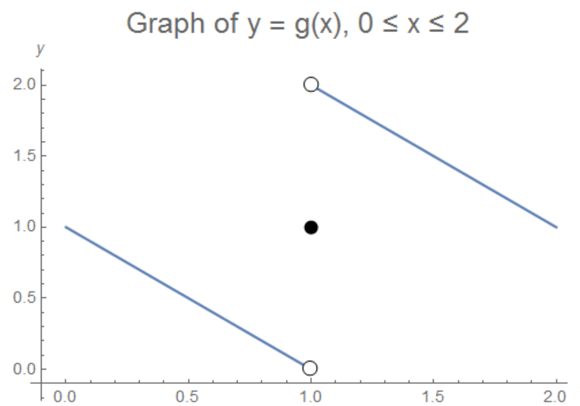
- On the open interval $(0, 1)$, the function $f(x) = x$ has neither an absolute minimum nor an absolute maximum: although the function takes any sufficiently small positive value, it does not attain the value 0 on the interval $(0, 1)$. Similarly, although the function takes values arbitrarily close to 1, it does not attain the value 1 on the interval $(0, 1)$.





- On the infinite interval $[1, \infty)$, the function $f(x) = 1/x$ has an absolute maximum value of x occurring at $x = 1$, but has no absolute minimum. As $x \to \infty$, the function approaches the value 0, but there is no real number x for which $f(x)$ is actually equal to 0.
- The function $f(x) = \begin{cases} 2x - x^2 & \text{for } 0 \le x < 1 \\ 1/2 & \text{for } x = 1 \\ 2x - x^2 & \text{for } 1 < x \le 2 \end{cases}$ on the closed interval $[0, 2]$ has the absolute minimum value of 0 occurring at $x = 0$ and $x = 2$, but has no absolute maximum because the function never attains the value 1 anywhere in the interval $[0, 2]$.



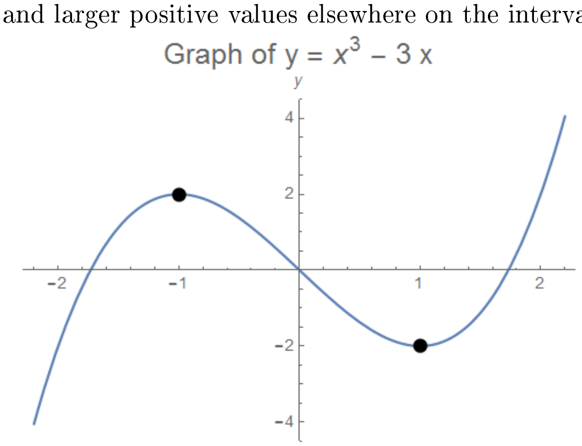


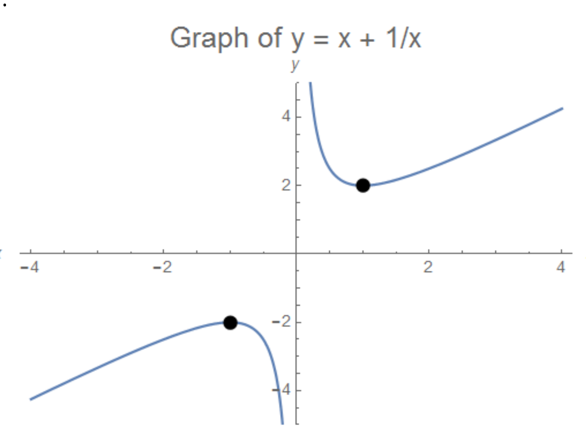
- The function $g(x) = \begin{cases} 1-x & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \\ 3-x & \text{for } 1 < x \le 2 \end{cases}$ on the closed interval $[0,2]$ has no absolute minimum or maximum on the interval, since it approaches the values 0 and 2 arbitrarily closely but never attains them.

- From these examples, it seems that in order to ensure that f has an absolute minimum and maximum on an interval I, we should require f to be continuous and make I contain its endpoints (i.e., require I to be a finite, closed interval). In fact, these two conditions suffice:
- Theorem (Extreme Value): If $f(x)$ is continuous on the closed interval $[a,b]$, then it attains its absolute maximum and absolute minimum on that interval. In other words, there exist real numbers m (the minimum) and M (the maximum) such that $m \le f(x) \le M$ on $[a,b]$, and real numbers c and d in $[a,b]$ such that $f(c) = m$ and $f(d) = M$.
  - The technical details of the proof of this theorem are not terribly enlightening, and it relies on a technical property of the real numbers known as the least upper bound axiom, so we will omit the details.

3.1.2 Local Minimum and Maximum Values, Critical Numbers

- There is another flavor of maximum / minimum point for us to analyze:
- Definition: For any function $f(x)$ and any value c, we say f has a local maximum at $x = c$ if $f(x) \le f(c)$ for all x in some open interval containing c. We say f has a local minimum at $x = c$ if $f(x) \ge f(c)$ for all x in some open interval containing c.
  - In contrast to an absolute maximum, being a local maximum only requires that f be smaller "nearby". Likewise, being a local minimum only requires that f be larger "nearby".
  - Example: The function $f(x) = x^3 - 3x$ has a local maximum at $x = 1$ and a local minimum at $x = -1$. Neither point is an absolute minimum or maximum on the interval $[-2,2]$, because f takes larger negative and larger positive values elsewhere on the interval.

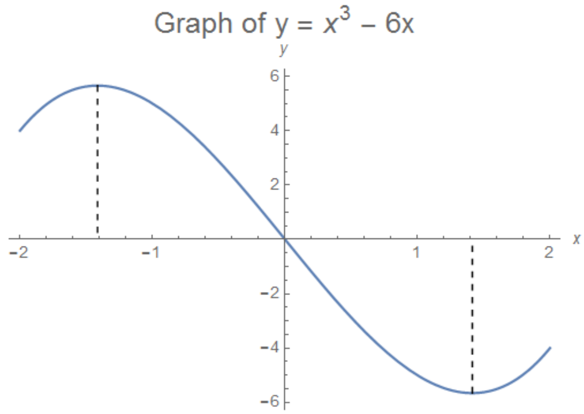


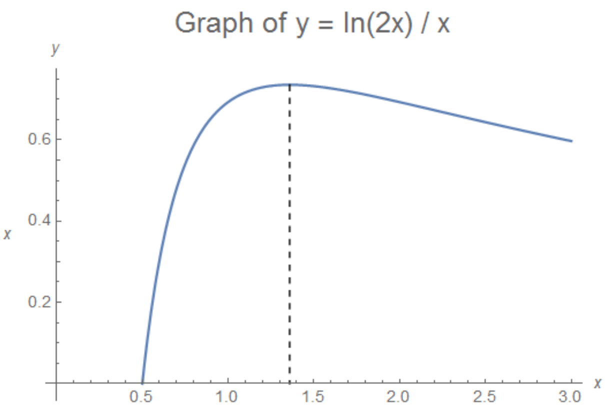


- Example: The function $g(x) = x + 1/x$ has a local maximum at $x = -1$ and a local minimum at $x = 1$. Neither point is an absolute minimum or maximum on the interval $[-4, 4]$.

- We can often visually identify the locations where a function seems to take local minimum and maximum values by looking at a graph. However, this procedure is not rigorous, and will not generally give exact answers.

  - Example: By studying the graph, it appears that $f(x) = x^3 - 6x$ has a local maximum at roughly $x \approx -1.4$, and a local minimum at roughly $x \approx 1.4$. (In fact the maximum is at $x = \sqrt{2}$ and the minimum is at $x = -\sqrt{2}$.)





  - Example: By studying the graph, it appears that the function $g(x) = \ln(2x)/x$ has a local maximum (and also absolute maximum) value occurring roughly at $x \approx 1.35$. (In fact, the exact location of the local maximum is $x = e/2$.)

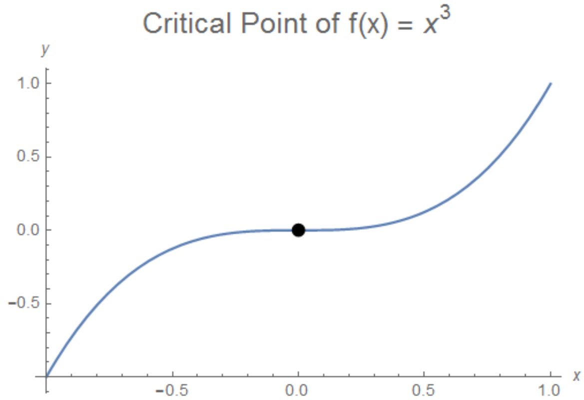
- Notice, in the examples above, that the tangent line to the curve $y = f(x)$ at a local minimum or maximum is always horizontal. This is not an accident:

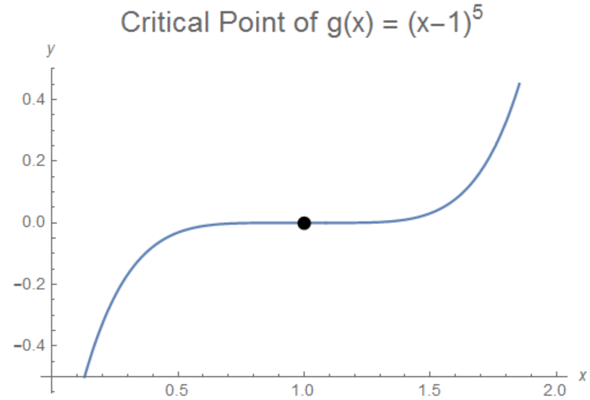
- Theorem (Fermat): If $f(x)$ is differentiable at $x = c$ and f has a local maximum or minimum at $x = c$, then $f'(c) = 0$.

  - Proof: Suppose first that f has a local minimum at $x = c$. We will show that $f'(c)$ is both at least zero and at most zero.
  - By definition, if f has a local maximum at $x = c$ then there is some open interval around c such that $f(c) \le f(x)$ for all x in that interval.
  - Since f is differentiable at $x = c$, the two-sided limit $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists. So in particular, both of the one-sided limits are equal to $f'(c)$.
  - Now $f'(c) = \lim_{x \to c+} \dfrac{f(x) - f(c)}{x - c}$, and since $f(c) \le f(x)$ and $c < x$ for all x in the limit, we see that $f'(c)$ is equal to a limit of nonnegative numbers, so by the inequality rule for limits we obtain $f'(c) \ge 0$.
  - In the same way, we also have $f'(c) = \lim_{x \to c-} \dfrac{f(x) - f(c)}{x - c}$, and since $f(c) \le f(x)$ and $x < c$ for all x in the limit, we see that $f'(c)$ is equal to a limit of nonpositive numbers, so by the inequality rule for limits we obtain $f'(c) \le 0$.
  - Combining the two inequalities $f'(c) \le 0$ and $f'(c) \ge 0$ shows that $f'(c) = 0$ as claimed.
  - Finally, in the event f has a local maximum at $x = c$, we may apply the argument above to $g(x) = -f(x)$ (noting that g has a maximum at $x = c$) to conclude that $g'(c) = 0$, and thus $f'(c) = -g'(c) = 0$ as well.

- Note, importantly, that the converse of Fermat's Theorem is *not true*! Even if $f'(c) = 0$, that does not necessarily mean that there is a local minimum or local maximum at $x = c$.

  - Example: The function $f(x) = x^3$ has $f'(0) = 0$, but 0 is neither a local maximum nor a local minimum for f.





- Example: The function $f(x) = (x-1)^5$ has $f'(1) = 0$, but 1 is neither a local maximum nor a local minimum for f.

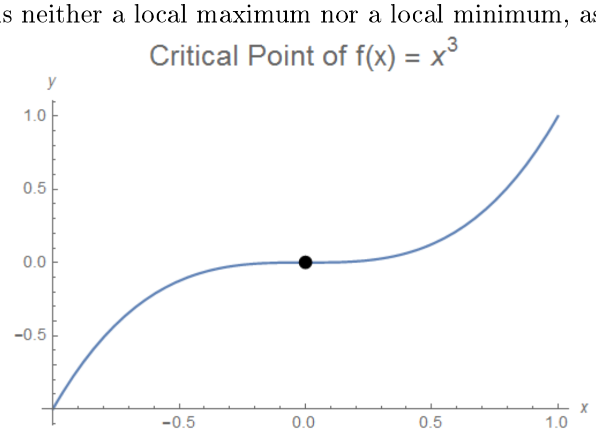
- Since we are interested in looking for minimum and maximum points, we give a name to these places where a minimum or maximum could potentially occur:

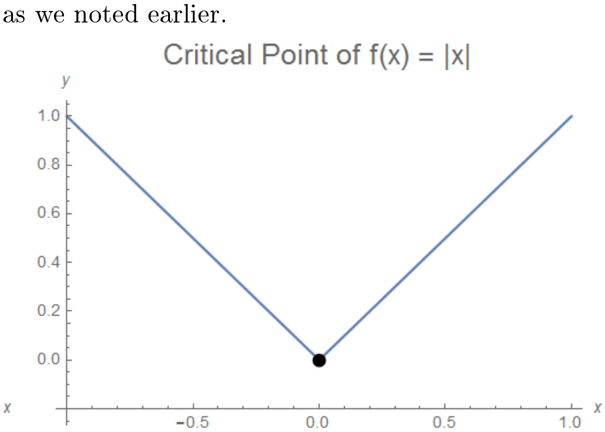
- Definition: A critical number of a function $f(x)$ is a value of x for which $f'(x) = 0$ or $f'(x)$ is undefined. If $x = a$ is a critical number, we say the point $(a, f(a))$ is a critical point.

  - Example: For $f(x) = x^2$, because $f'(0) = 0$, we say 0 is a critical number of f, and $(0, 0)$ is a critical point for f.
  - Notation: The terms "critical point" and "critical value" are often used interchangeably with "critical number". We will reserve "critical point" to refer to a point (x, y) on a graph where x is a critical number. The distinction is not particularly important, because the concepts are so closely related, but it can initially be confusing to switch freely from discussing critical numbers (representing x-coordinates) to discussing critical points (representing points on graphs).

- To emphasize, critical points are locations where f *could* have a local minimum or maximum. However, critical points are not *necessarily* either a minimum or a maximum. It is also very important to include locations where f' is undefined, in addition to places where f' is zero.

  - Example: The function $f(x) = x^3$ has a critical point at the origin since $f'(0) = 0$. But the critical point is neither a local maximum nor a local minimum, as we noted earlier.





  - Example: The function $f(x) = |x|$ has a critical point at the origin since $f'(0)$ is not defined. The critical point is a local (and absolute) minimum, since $|x| \geq 0$ for all x.

- Example: Find the critical numbers of $p(x) = x^5 - 5x$.

  - Because $p'(x) = 5x^4 - 5$ is always defined, we need only find the values for which $p'(x) = 0$.
  - We must solve $5x^4 - 5 = 0$, or equivalently $5(x^4 - 1) = 0$. Factoring yields $5(x-1)(x+1)(x^2+1) = 0$, so we get the two (real) solutions $x = 1$ and $x = -1$.

- Thus, the critical numbers of p are $x = \boxed{-1, 1}$.

- Example: Find the critical numbers of $h(x) = |3x - 2|$.

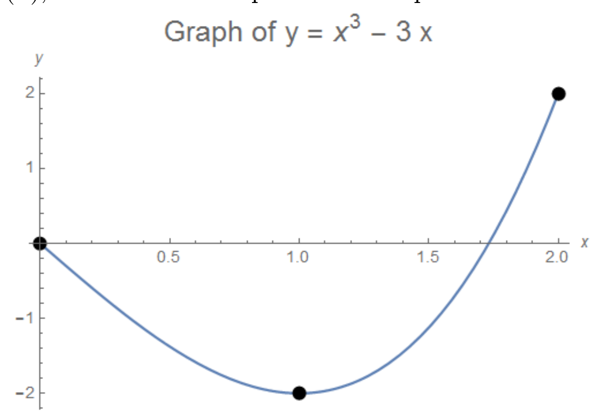
  - Since $h(x) = \begin{cases} 3x-2 & \text{for } x \geq 2/3 \\ 2-3x & \text{for } x < 2/3 \end{cases}$, we see that $h'(x) = \begin{cases} 3 & \text{for } x > 2/3 \\ -3 & \text{for } x < 2/3 \end{cases}$ and that $h'(2/3)$ is undefined.
  - Since h' is never zero, we conclude that $x = \boxed{2/3}$ is the only critical number of h.

- From our discussion above, if f is any continuous function defined on the interval $[a, b]$, the absolute minimum and absolute maximum values of f are each attained either at a critical number of f, or at one of the endpoints of the interval $[a, b]$.

  - Therefore, to find the minimum and maximum values of f on the interval, we need only make a list of the values of f at all of the critical numbers in the interval, along with the values of f at the endpoints a and b. The absolute maximum is the largest value on the list, while the absolute minimum is the smallest.

- Example: Find the absolute minimum and maximum values of $p(x) = x^3 - 3x$ on the interval $[0, 2]$.

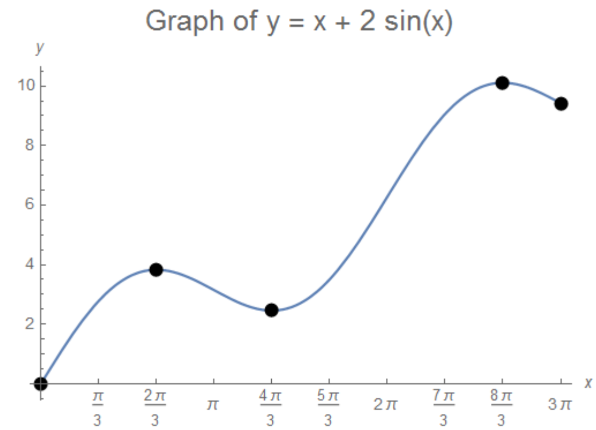
  - First, we make a list of the critical numbers of $p(x)$ inside the interval. Since $p'(x) = 3x^2 - 3$ is always defined, the only critical numbers occur when $p'(x) = 0$.
  - To solve $3x^2 - 3 = 0$, we can simply factor to get $3(x-1)(x+1) = 0$, so the critical numbers are $x = -1$ and $x = 1$.
  - The only critical number inside the interval is $x = 1$, so (including the two endpoints) we have 3 potential locations for the min and max: 0, 1, 2.
  - We compute $p(0) = 0$, $p(1) = -2$, and $p(2) = 2$.
  - Therefore, on $[0, 2]$, the absolute minimum of p is $\boxed{-2}$ (occurring at $x = 1$) and the absolute maximum is $\boxed{2}$ (occurring at $x = 2$).
  - Here is the graph of $y = p(x)$, with the critical point and endpoints marked:



- Example: Find the absolute minimum and maximum values of $f(x) = x + 2\sin(x)$ on the interval $[0, 3\pi]$.

  - First, we make a list of the critical numbers of $f(x)$ inside the interval. Since $f'(x) = 1 + 2\cos(x)$ is always defined, the only critical numbers occur when $f'(x) = 0$.
  - Since $f'(x) = 0$ is equivalent to $\cos(x) = -\dfrac{1}{2}$, we obtain three critical numbers in $[0, 3\pi]$: $x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{8\pi}{3}$.
  - Including the two endpoints, we have 5 potential locations for the min and max: $0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{8\pi}{3}, 3\pi$.
  - We compute $f(0) = 0$, $f(\frac{2\pi}{3}) = \frac{2\pi}{3} + \sqrt{3} \approx 3.826$, $f(\frac{4\pi}{3}) = \frac{4\pi}{3} - \sqrt{3} \approx 2.457$, $f(\frac{8\pi}{3}) = \frac{8\pi}{3} + \sqrt{3} \approx 10.110$, and $f(3\pi) = 3\pi \approx 9.424$.

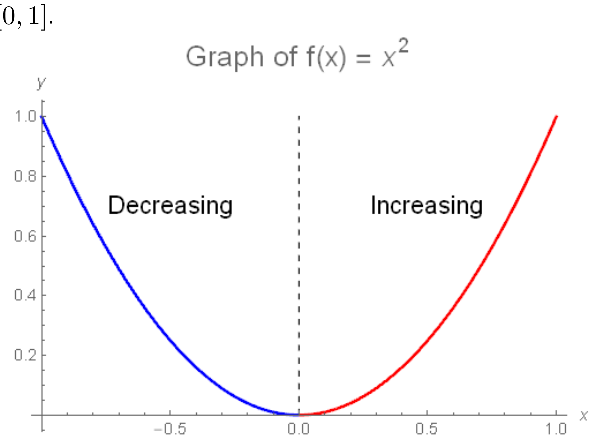
- Therefore, on $[0, 3\pi]$, the absolute minimum of f is $\boxed{0}$ (occurring at $x = 0$) and the absolute maximum is $\boxed{\dfrac{8\pi}{3} + \sqrt{3}}$ (occurring at $x = \frac{8\pi}{3}$).
- Here is the graph of $y = f(x)$, with the critical point and endpoints marked:

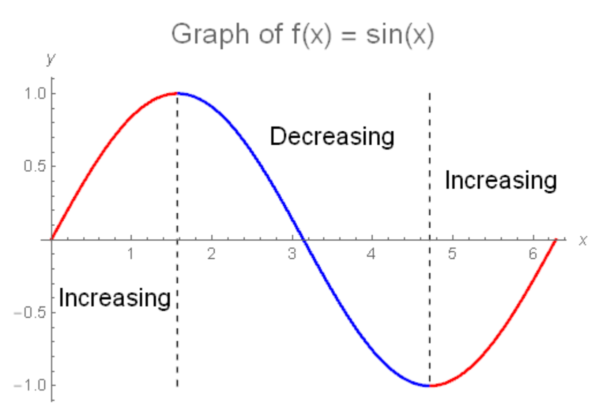


- We would also like to be able to decide when a critical number of f is a local minimum or local maximum. In order to do this most efficiently, however, we must discuss another topic first.

3.2 Increasing and Decreasing Functions

- In the previous section, we studied the behavior of f at locations where the derivative f' was equal to zero. We now turn our attention to studying the behavior of f at points where f' is not zero: specifically, we will discuss the ramifications that the sign of the derivative (positive or negative) has on the behavior of f.
- By definition, the value of the derivative $f'(c)$ measures how fast the value of $f(x)$ is changing when $x = c$.
  - In particular, if $f'(c) > 0$ then, almost by definition, intuition suggests that f should be "increasing" near $x = c$: if we increase x a small amount from c, f should take a larger value, and if we decrease x slightly, then the value of f should also decrease.
  - Inversely, if $f'(c) < 0$, f should be "decreasing" near $x = c$: if we increase x a small amount from c, f should decrease, and if we decrease x, then f should increase.
- In order to justify this intuitive sense rigorously we need to appeal to a few theorems, but first we need to give a better definition of what it means for a function to be increasing or decreasing:
- <u>Definitions</u>: If f is a function defined on an interval I, we f is <u>increasing</u> on I if $f(a) < f(b)$ for all $a < b$ in I. We say f is <u>decreasing</u> on I if $f(a) > f(b)$ for all $a < b$ in I.
  - <u>Example</u>: The function $f(x) = x^2$ is decreasing on the interval $[-1, 0]$ and increasing on the interval $[0, 1]$.



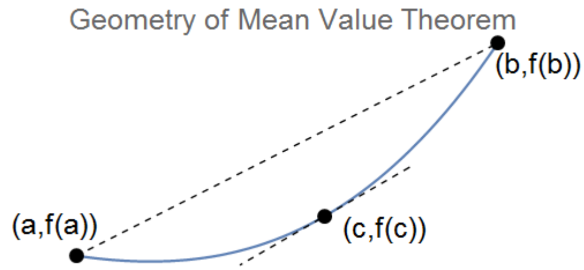


  - <u>Example</u>: The function $f(x) = \sin(x)$ is increasing on $[0, \pi/2]$ and $[3\pi/2, 2\pi]$ and decreasing on $[\pi/2, 3\pi/2]$.

3.2.1 Rolle's Theorem and the Mean Value Theorem

- We begin our analysis of increasing and decreasing functions with the following theorem, which will initially seem to be completely unrelated:
- Theorem (Rolle's Theorem): If $f(x)$ is a continuous function on $[a, b]$ which is differentiable on (a, b) and satisfies $f(a) = f(b)$, then there exists some point c in (a, b) for which $f'(c) = 0$.
  - Proof: By the Extreme Value Theorem, we know that $f(x)$ will attain its absolute maximum and absolute minimum somewhere on the interval $[a, b]$.
  - If the maximum is not at an endpoint, say at $x = c$, then by Fermat's Theorem we obtain that $f'(c) = 0$ and so we can take this maximum as the value of c.
  - Similarly, if the minimum is not at an endpoint, then we can use the x-coordinate of the minimum as the value of c.
  - Now if both the absolute maximum and minimum of the function $f(x)$ occur at the endpoints, then because $f(a) = f(b)$ the only possibility is for $f(x)$ to be constant. But in that case, we can take any point c in the interval we want, and $f'(c)$ will be zero.
- Corollary: If f is differentiable everywhere, then between any two zeroes of f there must be a zero of f'.
  - Proof: If $f(a) = 0$ and $f(b) = 0$, apply Rolle's Theorem to the interval $[a, b]$: there is necessarily a c in (a, b) with $f'(c) = 0$.
  - Thus, between the two zeroes a and b of f, there is a zero c of f'.
- We can use Rolle's Theorem (and the corollary above) to establish an upper bound on the number of real roots of a differentiable function.
  - When we combine Rolle's Theorem with appropriate use of the Intermediate Value Theorem to show the existence of real roots, we can often find the exact number of roots.
- Example: Show that the function $g(x) = x^3 + x - 1$ has at least one real root, and then show that g cannot have two real roots.
  - Notice that g is continuous, and $g(0) = -1$ while $g(1) = 1$. So by the Intermediate Value Theorem, we conclude that g must have a root somewhere in the interval $(0, 1)$.
  - Now suppose that g had another real root: call the two real roots a and b.
  - By the corollary to Rolle's Theorem, g' then would have to be zero somewhere between a and b.
  - But $g'(x) = 3x^2 + 1$ is never zero. This is an impossibility, meaning that g could not have two real roots.
  - Therefore, g must have exactly one real root.
- Example: Show that the polynomial $p(x) = x^7 - 7x + 1$ has exactly three real roots.
  - We use the Intermediate Value Theorem to show the existence of 3 roots, and then Rolle's Theorem (or more precisely, the corollary given above) to show that there cannot be more than 3 roots.
  - We compute $p(-2) = -114$, $p(-1) = 7$, $p(1) = -5$, and $p(2) = 115$, so p has a root on each of the intervals $(-2, -1)$, $(-1, 1)$, and $(1, 2)$, meaning it has at least 3 roots.
  - Now suppose that p had 4 or more roots, say $a < b < c < d$. Then p' would have to have a zero on each of the intervals (a, b), (b, c), and (c, d) so p' would have at least 3 roots.
  - But $p'(x) = 7x^6 - 7 = 7(x^6 - 1)$ only has two roots, $x = 1$ and $x = -1$, as can be seen by factoring or graphing. This is impossible, so p must have exactly 3 roots.
- Returning to our discussion of increasing and decreasing functions, we can use Rolle's Theorem to prove a more general result:
- Theorem (Mean Value Theorem): If $f(x)$ is a continuous function on $[a, b]$ which is differentiable on (a, b), then there exists some point c in (a, b) for which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

- Equivalently, this means there is a point c in the interval for which the instantaneous rate of change of $f(x)$ at $x = c$ (the value $f'(c)$) is equal to the average rate of change on $[a, b]$ (the value $\frac{f(b)-f(a)}{b-a}$).
- Another restatement: there is a point c in the interval (a, b) such that the slope of the tangent line to $y = f(x)$ at $x = c$ is equal to the slope of the secant line joining $(a, f(a))$ and $(b, f(b))$:



- Intuitively, from the picture, if we imagine "sliding" the secant line vertically, we will eventually hit a point where it is tangent to the graph of $y = f(x)$.
- Proof: The trick is to define a new function $g(x)$ to which we can apply Rolle's Theorem. The function we use is $g(x) = f(x) - x \cdot \dfrac{f(b) - f(a)}{b - a}$.
- Since $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), so is $g(x)$.
- Also, $g(b) - g(a) = [f(b) - f(a)] - (b - a) \cdot \dfrac{f(b) - f(a)}{b - a} = [f(b) - f(a)] - [f(b) - f(a)] = 0$.
- So we can apply Rolle's Theorem to $g(x)$, which says that there is a c in (a, b) for which $g'(c) = 0$.
- But this means $0 = g'(c) = f'(c) - \dfrac{f(b) - f(a)}{b - a}$, so that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ as we wanted.

3.2.2 Classification of Increasing and Decreasing Behavior

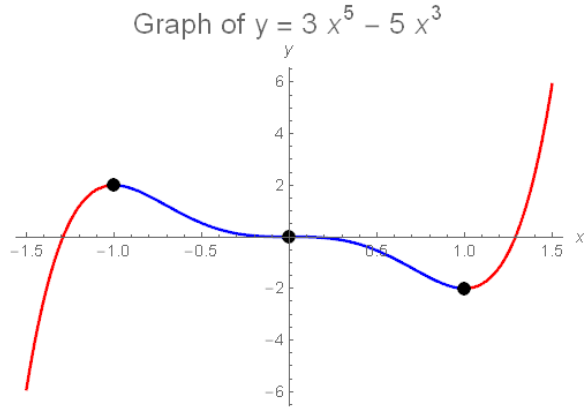
- Using the Mean Value Theorem, we can make very precise our intuitive ideas about the relationship between the sign of f' and the increasing or decreasing behavior of f.
- Theorem (Increasing and Decreasing Functions): If $f(x)$ is differentiable on an interval I, and $f'(x) > 0$ on for all x in I, then $f(x)$ is increasing on I. If $f'(x) < 0$ for all x in I, then $f(x)$ is decreasing on I.
  - Proof: First suppose that $f'(x) > 0$: we then want to show that for any $a < b$ in I, it is true that $f(a) < f(b)$.
  - By the Mean Value Theorem applied to f on $[a, b]$, there exists c in (a, b) with $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
  - By assumption $f'(x) > 0$ everywhere in I, so $f'(c) > 0$. Also, $b > a$ so $b - a$ is positive.
  - Therefore $f(b) - f(a)$ must be positive also, so $f(a) < f(b)$ as desired.
  - The argument when $f'(x) < 0$ is almost identical: applying the Mean Value Theorem on $[a, b]$ again yields some c in (a, b) with $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, but now $f'(c) < 0$ implies that $f(b) - f(a)$ must be negative, so then $f(b) < f(a)$.
- By the theorem, f is increasing whenever $f' > 0$, and f is decreasing whenever $f' < 0$.
  - Provided that f' is continuous, to determine where f' is positive and negative, we may list all of the critical numbers (the places where f' is zero or undefined) and then plug in a "test point" in each interval to determine the sign of f' on that interval.
  - By the Intermediate Value Theorem, we are then guaranteed that f' cannot change sign anywhere else, so f' has the same sign on each interval.
- Example: For $f(x) = x^3 - 3x$, determine where f is increasing and where f is decreasing.

- Since $f'(x) = 3x^2 - 3$ is continuous, we may use test points. First, we identify the critical numbers of f: setting $3x^2 - 3 = 0$ yields $3(x-1)(x+1) = 0$ so the critical numbers are $x = -1$ and $x = 1$.
- We therefore have three intervals to consider: $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. We choose a test point in each interval, and determine whether f' is positive or negative there: then f' necessarily has the same sign in the entire interval.
- On $(-\infty, -1)$ we can choose the test point $x = -2$: since $f'(-2) = 9 > 0$, we see that $f' > 0$ on this interval.
- On $(-1, 1)$ we can choose the test point $x = 0$: since $f'(0) = -3 < 0$, we see that $f' < 0$ here.
- Finally, on $(1, \infty)$ we can choose the test point $x = 2$: since $f'(2) = 9 > 0$, we see that $f' > 0$ here.
- We can summarize this information with a sign diagram for f' : $\oplus \underset{-1}{|} \ominus \underset{1}{|} \oplus$.
- By the discussion above, f is therefore $\boxed{\text{increasing on } (-\infty, -1) \text{ and } (1, \infty)}$ and $\boxed{\text{decreasing on } (-1, 1)}$.

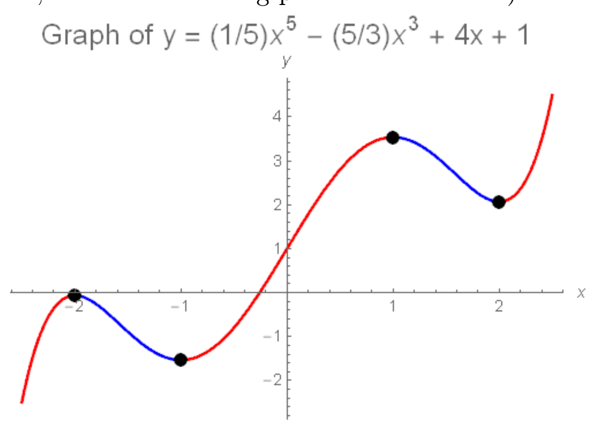
3.2.3 The First Derivative Test

- We can also use our analysis of increasing and decreasing behaviors to give a procedure for determining whether a critical number is a local minimum, local maximum, or neither.
- Theorem (First Derivative Test): Suppose c is a critical number of f. If f' changes sign from negative to positive at c , then f has a local minimum at c. If f' changes sign from positive to negative at c, then f has a local maximum at c. Finally, if f' does not change sign at c, then c is neither a minimum nor a maximum.
  - When we say "f' changes sign from negative to positive at c", this means that $f' < 0$ on some interval (a, c) for some $a < c$ and that $f' > 0$ on some other interval (c, b) for some $c < b$.
  - In other words, f' is negative for numbers slightly smaller than c, and f' is positive for numbers slightly larger than c.
  - Proof: First suppose that f' changes sign from negative to positive at c. This means that $f' < 0$ on some interval (a, c) for some $a < c$, and that $f' > 0$ on some other interval (c, b) for some $c < b$.
  - By the theorem on increasing and decreasing functions, we can then conclude that $f(x) < f(c)$ for all $a < x < c$, and also that $f(c) > f(y)$ for all $c < y < b$. Therefore, $f(c) \geq f(z)$ for all $a < z < b$, meaning that c is a local maximum of f.
  - The proof where f' changes sign from positive to negative is the same, except with the appropriate inequalities flipped. Finally, if f' does not change sign, then f is either increasing or decreasing on an interval around c, meaning that f has neither a minimum nor a maximum at c.
- For reasonable functions, the First Derivative Test says that we can simply read off the type of critical point from the f' sign diagram.
- Example: For $f(x) = 3x^5 - 5x^3$, find and classify the critical numbers of f (local minimum, local maximum, or neither), and determine where f is increasing and where f is decreasing.
  - We have $f'(x) = 15x^4 - 15x^2$ which is always defined. Setting $f' = 0$ and factoring yields $15x^2(x-1)(x+1) = 0$, so there are three critical numbers: $x = \boxed{-1,\ 0,\ 1}$.
  - Next we construct the sign diagram for f': using test points $x = -2$, $-1/2$, $1/2$, and 2 yields the sign diagram f' : $\oplus \underset{-1}{|} \ominus \underset{0}{|} \ominus \underset{1}{|} \oplus$.
  - Therefore, f is $\boxed{\text{increasing on } (-\infty, -1) \text{ and } (1, \infty)}$ and $\boxed{\text{decreasing on } (-1, 0) \text{ and } (0, 1)}$.
  - Furthermore, at $x = -1$ we see that f' switches from positive to negative, meaning that f switches from increasing to decreasing, so that $\boxed{-1 \text{ is a local maximum}}$.
  - At $x = 0$, f' does not change sign, so $\boxed{0 \text{ is neither a minimum nor a maximum}}$.
  - Finally, at $x = -1$ we see that f' switches from negative to positive, meaning that f switches from decreasing to increasing, so that $\boxed{1 \text{ is a local minimum}}$.

◦ Here is the graph of $y = f(x)$, illustrating what we have found (the critical points are marked, the increasing portions are in red, and the decreasing portions are in blue):



- Example: For $p(x) = \frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x + 1$, find and classify the critical numbers of f (local minimum, local maximum, or neither), and determine where f is increasing and where f is decreasing.
  - ◦ We have $p'(x) = x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4) = (x - 1)(x + 1)(x - 2)(x + 2)$.
  - ◦ Thus, the critical numbers are $\boxed{x = -2, -1, 1, 2}$, since p' is defined everywhere.
  - ◦ Plugging in test points (e.g., $x = -3$, -1.5, 0, 1.5, 3) yields the sign diagram $\oplus \underset{-2}{|} \ominus \underset{-1}{|} \oplus \underset{1}{|} \ominus \underset{2}{|} \oplus$ for p'.
  - ◦ Thus, p is $\boxed{\text{increasing on } (-\infty, -2),\ (-1, 1), \text{ and } (2, \infty)}$, and $\boxed{\text{decreasing on } (-2, -1) \text{ and } (1, 2)}$.
  - ◦ Also, $\boxed{x = -1 \text{ and } 2 \text{ are local minima}}$ since p switches from decreasing to increasing at both locations, and $\boxed{x = -2 \text{ and } 1 \text{ are local maxima}}$ since p switches from increasing to decreasing at both locations.
  - ◦ Here is the graph of $y = p(x)$, illustrating what we have found (the critical points are marked, the increasing portions are in red, and the decreasing portions are in blue):



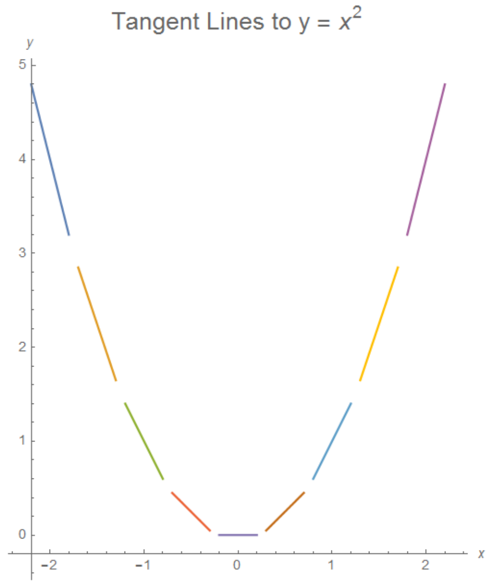
3.3 Concavity, Graphing With Calculus

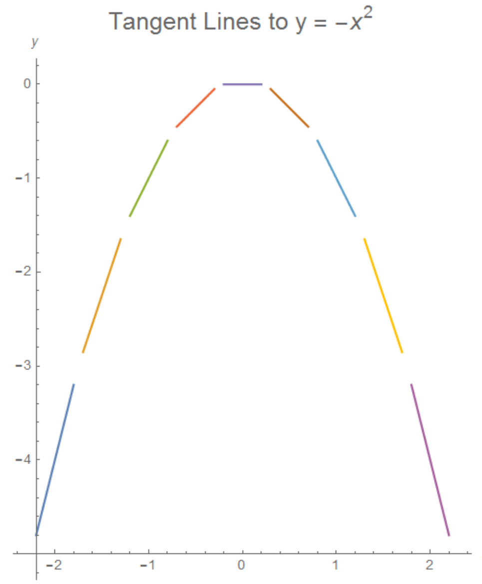
- We have already discussed how to use the first derivative f' of a function f to analyze its behavior: we now turn our attention to studying what the second derivative f'' can tell us. Then we will explain how to use all of this information to draw an accurate graph of $y = f(x)$.

3.3.1 Concavity, Inflection Points, and the Second Derivative

- By definition, f'' represents the rate of change of the derivative f'.

- Thus, the statement that $f'' > 0$ on an interval is equivalent to saying that the function f' is increasing on that interval.
- Because f' represents the slope of the tangent line to the graph of $y = f(x)$, the statement $f'' > 0$ says that the slopes of the tangent lines to the graph of $y = f(x)$ are increasing.
- We can see this quite clearly from the graph of the function $y = f(x)$ for $f(x) = x^2$, whose second derivative is $f''(x) = 2$: in this graph, the tangent slopes clearly increase as we move from left to right along the plot, producing an overall "upward-opening" shape that rather closely matches the actual shape of the graph of $y = x^2$:





- By the same logic, if $f'' < 0$, then the tangent slopes to the graph of $y = f(x)$ should be decreasing, suggesting that the graph should have a "downward-opening" shape.
- This intuition is borne out by the graph (above) of $y = f(x)$ for $f(x) = -x^2$, whose second derivative is $f''(x) = -2$.

- Let us give a more precise definition of this interpretation of the behavior of f' (namely, whether it is increasing or decreasing):

- Definition: If f is a differentiable function defined on an interval I, we f is concave up on I if f' is increasing on I, and we say f is concave down on I if f' is decreasing on I.

- As we already worked out intuitively above, the concavity of a function is related to the sign of its second derivative:

- Theorem (Concavity of Functions): If $f(x)$ is twice differentiable on an interval I, and $f''(x) > 0$ for all x in I, then $f(x)$ is concave up on I. If $f''(x) < 0$ for all x in I, then $f(x)$ is concave down on I.
  - Proof: Let $g = f'$: then by our earlier theorem about increasing functions, if $g' > 0$ on I then g is increasing.
  - Rephrasing in terms of f: if $f'' > 0$ on I, then f' is increasing, or, equivalently, f is concave up.
  - If $g' < 0$ then g is decreasing, which is equivalent to saying that if $f'' < 0$ on I, then f' is decreasing, so f is concave down.

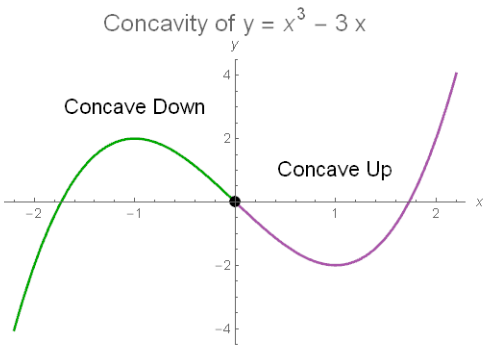
- The locations where f changes concavity have a special name:

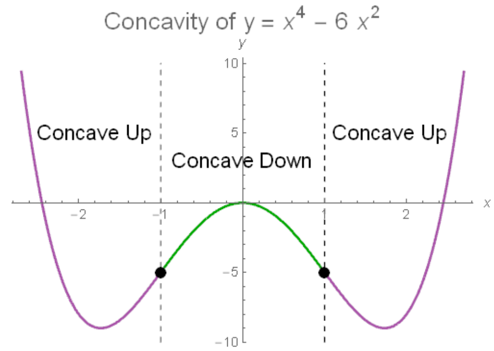
- Definition: If f changes concavity at a point, that point is called a point of inflection (or inflection point).
  - Points of inflection are sometimes called turning points: when drawing the graph, you will find that you must "turn" your hand when the curve passes through a point of inflection.
  - If f' exists everywhere, then the points of inflection are precisely the places where f'' changes sign, and the graph of $y = f(x)$ will "flatten out" at such inflection points.

- Remark: Like with critical numbers and critical points, we reserve the term "inflection point" for a point (x, y) on the graph of $y = f(x)$, and refer to the x-coordinates of inflection points as inflection numbers.

- Let us give a few examples of concavity and points of inflection:

  - Example: Because $f(x) = x^3 - 3x$ has $f''(x) = 6x$, we see that f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$, and has a point of inflection at $(0, 0)$.





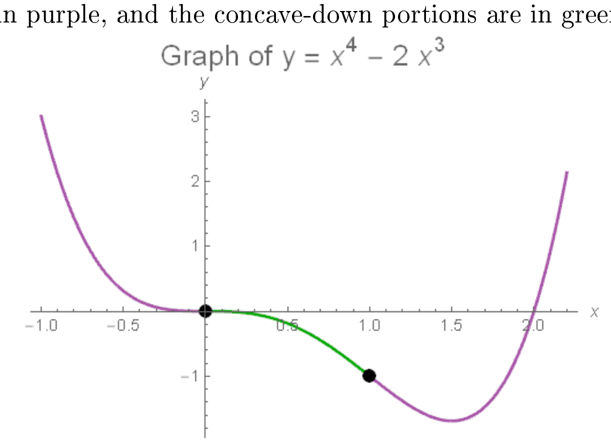
  - Example: Because $f(x) = x^4 - 6x^2$ has $f''(x) = 12(x^2 - 1)$, we see that f is concave up on $(-\infty, -1) \cup (1, \infty)$ and concave down on $(-1, 1)$. Furthermore, because f'' changes sign at $x = -1$ and $x = 1$, it has points of inflection at $(-1, -5)$ and $(1, -5)$.

- If f'' is continuous, then to find points of inflection and study concavity, we can make a sign diagram for f'' using test points in exactly the same way that we did for f'.

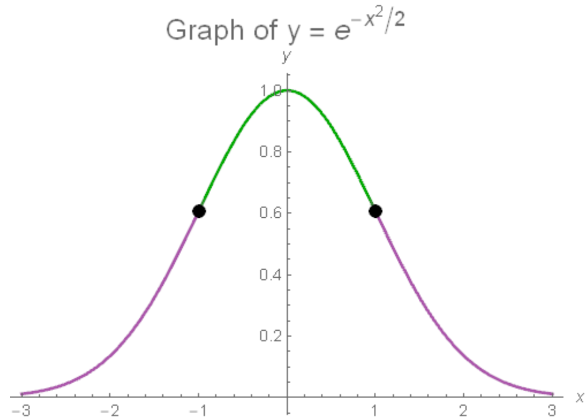
  - Explicitly, we first mark off all places where f'' is zero or undefined. By the continuity of f'', there are no other places where f'' could change sign.
  - We then plug in a test point in each interval to determine the sign of f'' there. The intervals where $f'' > 0$ are where f is concave up, the intervals where $f'' < 0$ are where f is concave down, and the places where f'' changes sign correspond to the inflection points.

- Example: For $p(x) = x^4 - 2x^3$, find the inflection points of f, the intervals where f is concave up, and the intervals where f is concave down.

  - We have $p'(x) = 4x^3 - 6x^2$ and $p''(x) = 12x^2 - 12x = 12x(x-1)$.
  - Thus, the potential points of inflection are $x = 0, 1$, since p'' is defined everywhere.
  - Plugging in test points (e.g., $x = -1$, 0.5, 2) yields the sign diagram $\oplus \underset{0}{|} \ominus \underset{1}{|} \oplus$ for p''.
  - Thus, p is $\boxed{\text{concave up on } (-\infty, 0), \text{ and } (1, \infty)}$, and $\boxed{\text{concave down on } (0, 1)}$.
  - Also, since p'' switches sign at $x = 0$ and at $x = 1$, we see that $\boxed{(0, 0) \text{ and } (1, -1) \text{ are points of inflection}}$ for p.
  - Here is the graph of $y = p(x)$, illustrating what we have found (the inflection points are marked, the concave-up portions are in purple, and the concave-down portions are in green):



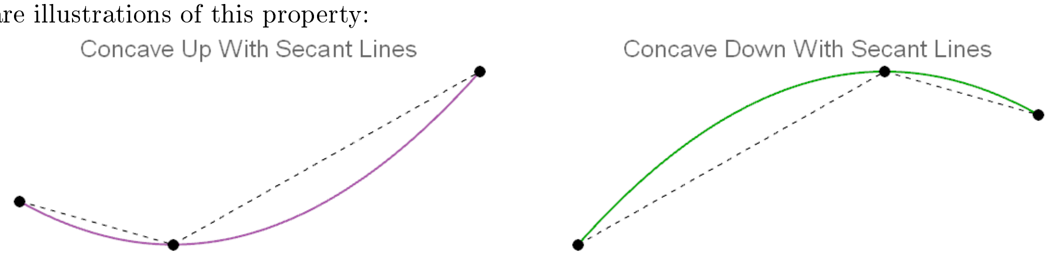
- Example: For $f(x) = e^{-x^2/2}$, find the inflection points of f, the intervals where f is concave up, and the intervals where f is concave down.
  - We have $f'(x) = -xe^{-x^2/2}$, and $f''(x) = -e^{-x^2/2} + (-x)(-x)e^{-x^2/2} = (x^2-1)e^{-x^2/2}$.
  - Since f'' is always defined, we see that the only points of inflection could occur when $f''(x) = 0$; namely, when $x = \pm 1$.
  - Plugging in test points (e.g., $x = -2, 0, 2$) yields the sign diagram $\oplus \underset{-1}{|} \ominus \underset{1}{|} \oplus$ for f''.
  - Thus, f is $\boxed{\text{concave up on } (-\infty, -1), \text{ and } (1, \infty)}$, and $\boxed{\text{concave down on } (-1, 1)}$.
  - Also, since f'' switches sign at $x = -1$ and at $x = 1$, we see that $\boxed{(-1, e^{-1/2}) \text{ and } (1, e^{-1/2}) \text{ are points of inflection}}$ for f.
  - Here is the graph of $y = f(x)$, illustrating what we have found (the inflection points are marked, the concave-up portions are in purple, and the concave-down portions are in green):



  - Remark: The curve analyzed above is (up to a scaling factor) the famous Gaussian normal distribution, commonly called the "bell curve". It shows up very frequently in statistics.

3.3.2 Geometric Properties of Concavity, the Second Derivative Test

- There are some other useful geometric interpretations of concavity. Here are two of them:
- Theorem (Concavity and Secant Lines): A function is concave up on an interval if it lies below its secant lines: in other words, if the portion of the graph between $x = a$ and $x = b$ lies below the line joining $(a, f(a))$ and $(b, f(b))$. A function is concave down if it lies above its secant lines.
  - Here are illustrations of this property:

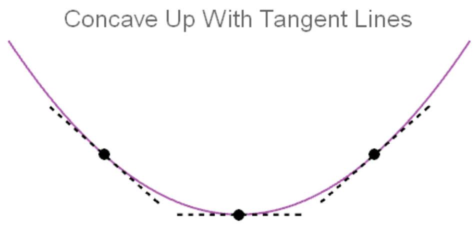


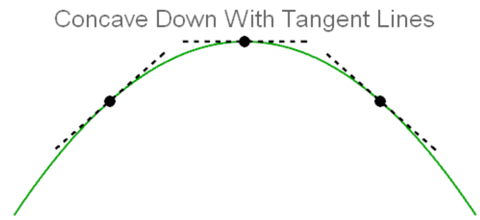
  - Proof: Suppose first that f is concave up.
  - Observe that the given statement is equivalent to saying that for any t with $a < t < b$, one has $f(t) < \dfrac{b-t}{b-a}f(a) + \dfrac{t-a}{b-a}f(b)$. (The right-hand term is the y-coordinate of the point with x-coordinate t on the line joining $(a, f(a))$ and $(b, f(b))$.)
  - This inequality can be rearranged into the equivalent form $\dfrac{f(t)-f(a)}{t-a} < \dfrac{f(b)-f(t)}{b-t}$, which holds by the Mean Value Theorem: the left term is equal to $f'(c)$ for some c in (a, t), and the right term is equal

to $f'(d)$ for some d in (t, b): then because $f''(x) > 0$, we know that f' is increasing, so $f'(c) < f'(d)$, as desired.

- For concave-down functions the argument is similar, except with the appropriate inequalities flipped.

- Theorem (Concavity and Tangent Lines): A function is concave up on an interval if it lies above its tangent lines (except for the points of tangency), and a function is concave down on an interval if it lies below its tangent lines.
  - Here are illustrations of this property:





  - Proof: Suppose first that f is concave up. Observe that the given statement is equivalent to the statement that $f(t) > f(a) + f'(a) \cdot (t - a)$ for any $t \neq a$. (The right-hand term is the y-coordinate of the point with x-coordinate t on the tangent line to $y = f(x)$ at $x = a$.)
  - For $t < a$, this inequality is equivalent to $\dfrac{f(t) - f(a)}{t - a} < f'(a)$, which holds by the Mean Value Theorem: the left term is equal to $f'(c)$ for some c in (t, a), and $f'(c) < f'(a)$ by the assumption that f' is increasing.
  - Similarly, if $t > a$, then the inequality is equivalent to $f'(a) < \dfrac{f(t) - f(a)}{t - a}$, which again holds by the Mean Value Theorem: the left term is equal to $f'(d)$ for some d in (a, t), and $f'(a) < f'(d)$ by the assumption that f' is increasing.
  - For concave-down functions the argument is similar, except with the appropriate inequalities flipped.

- As a final remark, there is a way to use the second derivative to determine whether a critical number is a local minimum or local maximum.
  - The method we previously discussed, of determining whether f changes sign at a critical number, will always work at least as well as this test does. (This test does have the advantage of occasionally requiring slightly less computation.) We include this method only for completeness.

- Theorem (Second Derivative Test): If f is a twice-differentiable function with c a critical number (so that $f'(c) = 0$), then c is a local maximum if $f''(c) < 0$ and c is a local minimum if $f''(c) > 0$.
  - Remark: If $f''(c) = 0$, the test yields no information. There could be a local minimum, local maximum, or neither.
  - Intuitively, if $f'' > 0$ then f is concave up, so any critical number must be a local minimum because the graph must open "upwards". Inversely, if $f'' < 0$ then f is concave down, so any critical number must be a local maximum because the graph must open "downwards".
  - Proof: First suppose $f''(c) > 0$ and $f'(c) = 0$: then f'' is positive on some open interval containing c.
  - By the theorem on concavity and tangent lines, the graph of $y = f(x)$ lies above the graph of its tangent line at $x = c$ on an open interval containing c.
  - But the tangent line is horizontal, so we immediately conclude that $f(x) \geq f(c)$ on an open interval containing c, which means that c is a local minimum.
  - If $f''(c) < 0$ and $f'(c) = 0$, then in a similar way we conclude that f lies below the graph of its horizontal tangent line at c, meaning $f(x) \leq f(c)$ and so c is a local maximum.

- Example: Use the Second Derivative Test to identify the critical numbers of $f(x) = x^3 - 12x$ as local minima or local maxima.
  - We have $f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2)$, so the critical numbers are $x = -2$ and $x = 2$.
  - Since $f''(x) = 6x$, we see $f''(-2) = -12 < 0$, meaning that $\boxed{-2 \text{ is a local maximum}}$, and $f''(2) = 12 > 0$, so $\boxed{2 \text{ is a local maximum}}$.

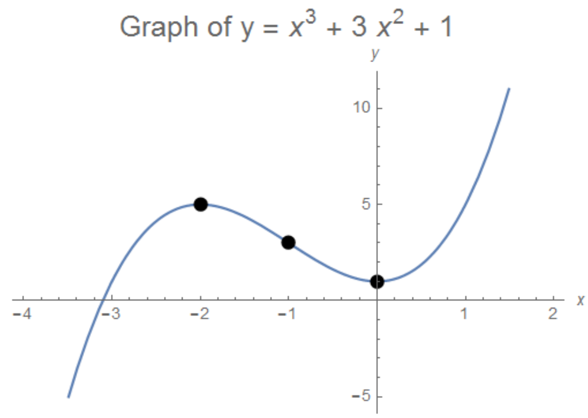
3.3.3 Graphing Functions Using Calculus

- By analyzing the first and second derivatives of a function f, we can obtain a great deal of information about f: we can determine the locations of critical points and inflection points, classify relative minima and maxima, and determine where the function is increasing, decreasing, concave up, and concave down.
- Using this information, we can plot graphs of functions more precisely (and usually, more easily) than the standard procedure of plugging many points into the function and joining them with a curve.
  - Generally speaking, the most interesting features of the graph of $y = f(x)$ are places near critical points or inflection points. We can identify these locations by studying the derivatives f' and f'' and determining where the values change sign.
  - Once we have identified the approximate locations of "interesting" features of the graph, we can fill in the portion of the graph between those points: we will know fairly accurately how the function behaves between its critical points and inflection points (i.e., whether it is increasing or decreasing, and whether it is concave up or concave down).
- If the graph of $y = f(x)$ approaches a line (in some manner), we call that line an <u>asymptote</u> of the graph. Specifically:
  - If $\lim_{x\to a+} f(x) = \infty$ or $-\infty$, or $\lim_{x\to a-} f(x) = \infty$ or $-\infty$, the line $x = a$ is called a <u>vertical asymptote</u> of $y = f(x)$. Typically, vertical asymptotes occur because of division by zero, e.g., $\frac{1}{x}$ at $x = 0$, but they can also occur for other functions like $\ln(x)$ at $x = 0$.
  - If $\lim_{x\to\infty} [f(x) - (Ax + B)] = 0$ (respectively, $\lim_{x\to-\infty} [f(x) - (Ax + B)] = 0$), then the line $y = Ax + B$ is called a <u>slanted asymptote</u> (or <u>horizontal asymptote</u> if $A = 0$) to the graph of $y = f(x)$ as $x \to \infty$ (respectively, as $x \to -\infty$).
- To identify all of the important features of a graph $y = f(x)$, we can follow these steps:
  - First, analyze f' to identify any local minima or maxima, and the intervals where f is increasing or decreasing.
    * Find all critical points by determining when $f'(x) = 0$, and when $f'(x)$ is undefined.
    * Mark all the critical points on a number line, and then in each interval between two critical points (as well as the infinite interval out to $-\infty$ and the interval out to $+\infty$), plug in a test value to f' to determine whether f' is positive or negative on that interval.
    * When $f' > 0$, f is increasing, and when $f' < 0$, f is decreasing.
    * Furthermore, at each critical point, if f' changes sign from positive to negative then f has a local maximum, and if f' changes sign from negative to positive then f has a local minimum. (If f' does not change sign, then f has neither a minimum nor a maximum.)
  - Next, analyze f'' to identify any points of inflection, and the intervals where f is concave up or concave down.
    * Find $f''(x)$ and set it equal to zero, to find all potential points of inflection.
    * Mark all potential points of inflection on a number line (along with all points where f'' is undefined), and then plug in test points to f'' to determine the sign of f'' on each interval.
    * When $f' > 0$, f is concave up, and when $f' < 0$, f is concave down.
    * When f'' changes sign, f has an inflection point.
  - Then, analyze the behavior of $f(x)$ for large x, and look for any asymptotes.
    * Find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$. If the limit as $x \to -\infty$ or $x \to \infty$ has a finite value L, then f has a horizontal asymptote $y = L$ as $x \to -\infty$ or $x \to \infty$ (as appropriate).
    * To find vertical asymptotes, search for values of c such that $\lim_{x\to c-} f(x)$ or $\lim_{x\to c+} f(x)$ (or both) are equal to ∞ or $-\infty$.

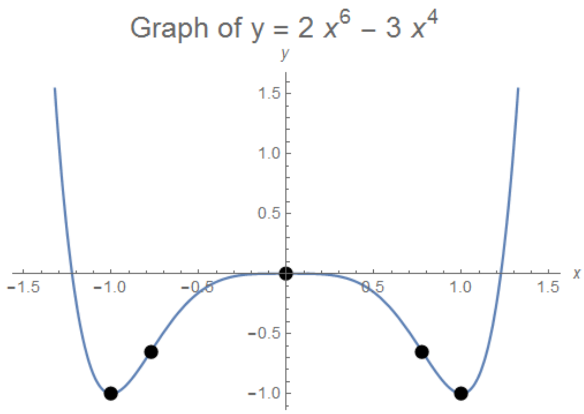
    * To find slanted asymptotes, first compute $\lim_{x\to\infty} \frac{f(x)}{x}$. If this first limit exists and is a finite nonzero number A, then compute the limit $\lim_{x\to\infty} [f(x) - Ax]$. If this second limit also exists and is a finite number B, then $y = Ax + B$ is a slanted asymptote as $x \to \infty$. Repeat the process with limits as $x \to -\infty$.

  - Finally, assemble all of the information (increasing/decreasing behavior, minima and maxima, concavity, points of inflection, behavior for large x, asymptotes) to draw the graph.
    * Begin by computing the coordinates of all of the critical and inflection points and plotting them accurately in the plane.
    * Next, on each of the x-intervals between consecutive pairs of the plotted points, identify the behavior of the function on that interval (increasing and concave up, increasing and concave down, decreasing and concave up, or decreasing and concave down) and draw an appropriately-shaped curve joining the two endpoints of the interval.
    * Finally, use the information about the behavior as $x \to -\infty$ and $x \to \infty$ to plot the behavior of f for the remaining values of x.

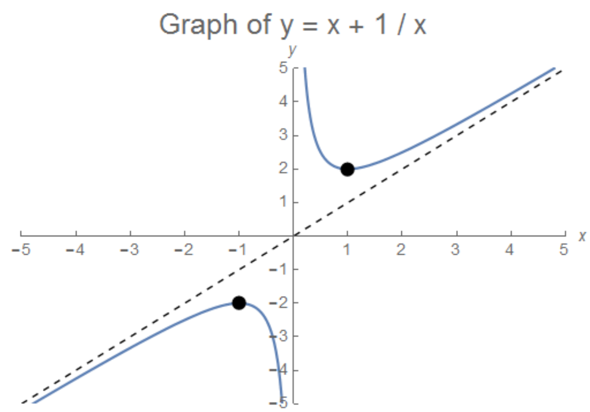
- <u>Example</u>: Let $f(x) = x^3 + 3x^2 + 1$. Find the critical points, local minima and maxima, points of inflection, and the intervals where f is increasing, decreasing, concave up, or concave down. Then sketch the graph of $y = f(x)$.
  - First, we analyze $f'(x) = 3x^2 + 6x = 3x(x + 2)$.
    * Setting $f'(x) = 0$ yields $3x(x + 2) = 0$, so we see that $\boxed{x = -2,\ 0 \text{ are critical points}}$.
    * Next, we draw the number line, mark off the two critical points, and then plug in test points (for example, we could use $x = -3$, $x = -1$, and $x = 1$) to see that the f' sign diagram looks like $\oplus \underset{-2}{|} \ominus \underset{0}{|} \oplus$. Alternatively, we could have used the factored form $f'(x) = 3x(x + 2)$ to make the sign diagram.
    * From this, we conclude that f is $\boxed{\text{increasing on } (-\infty, -2) \text{ and } (0, \infty)}$ and $\boxed{\text{decreasing on } (-2, 0)}$.
    * Also, since f' changes from positive to negative at $x = -2$, $\boxed{(-2, 5) \text{ is a local maximum}}$, and since f' changes from negative to positive at $x = 0$, $\boxed{(0, 1) \text{ is a local minimum}}$.
  - Next, we analyze $f''(x) = 6x + 6$.
    * Setting $f''(x) = 0$ yields $x = -1$.
    * We draw the number line, mark off $x = -1$, and then plug in two test points to see that the f'' sign diagram looks like $\ominus \underset{-1}{|} \oplus$.
    * From this, we conclude that f is $\boxed{\text{concave up on } (-1, \infty)}$ and $\boxed{\text{concave down on } (-\infty, -1)}$, and since f'' changes sign at $x = -1$ we see that $\boxed{(-1, 3) \text{ is a point of inflection}}$.
  - Since f is a polynomial of odd degree with positive leading coefficient, $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. There are also no asymptotes, again because f is a polynomial.
  - Now we can assemble the graph.
    * From $x = -\infty$, f is concave down and increasing from $-\infty$, until it hits a local max at $(-2, 5)$ and begins decreasing.
    * The function has a point of inflection at $(-1, 3)$, switching to concave up, and continues decreasing until it hits a local min at $(0, 1)$.
    * Then f increases and is concave up out to $x = +\infty$.
    * Here is the resulting graph (with the local extrema and point of inflection marked):



- Example: Let $h(x) = 2x^6 - 3x^4$. Find the critical numbers, local minima and maxima, points of inflection, intervals where f is increasing, decreasing, concave up, or concave down, the absolute minimum and maximum of h on the interval$-2 \le x \le 2$, and finally, sketch the graph of $y = h(x)$.
  - First, we analyze $h'(x) = 12x^5 - 12x^3 = 12x^3(x^2-1) = 12x^3(x+1)(x-1)$.
    - Setting $h'(x) = 0$ yields $12x^3(x+1)(x-1) = 0$, so we see that $\boxed{x = -1, 0, 1 \text{ are critical numbers}}$.
    - Next, we draw the number line, mark off the three critical points, and then plug in test points (for example, we could use $x = -2$, $x = -1/2$, $x = 1/2$, and $x = 2$) to see that the f' sign diagram looks like $\ominus \underset{-1}{|} \oplus \underset{0}{|} \ominus \underset{1}{|} \oplus$. Alternatively, we could have used the factorization $h'(x) = 12x^3(x+1)(x-1)$.
    - Thus, f is $\boxed{\text{increasing on } (-1,0) \text{ and } (1,\infty)}$ and $\boxed{\text{decreasing on } (-\infty,-1) \text{ and } (0,1)}$.
    - Also, since f' changes from positive to negative at $x = 0$, $\boxed{(0,0) \text{ is a local maximum}}$.
    - Since f' changes from negative to positive at $x = -1$ and $x = 1$, $\boxed{(-1,-1) \text{ and } (1,-1) \text{ are local minima}}$.
  - Next, we analyze $h''(x) = 12(5x^4 - 3x^2) = 12x^2(5x^2-3) = 60x^2(x - \sqrt{3/5})(x + \sqrt{3/5})$.
    - Setting $h''(x) = 0$ yields $x = 0, \pm\sqrt{3/5}$.
    - We draw the number line, mark off the three points, and then plug in test points to see that the f'' sign diagram looks like $\oplus \underset{-\sqrt{\frac{3}{5}}}{|} \ominus \underset{0}{|} \ominus \underset{\sqrt{\frac{3}{5}}}{|} \oplus$. (Again, we could also have used the factorization to help.)
    - Thus, f is $\boxed{\text{concave up on } (-\infty,-1) \text{ and } (1,\infty)}$ and $\boxed{\text{concave down on } (-1,-0) \text{ and } (0,1)}$, and since f'' changes sign at $x = \pm\sqrt{3/5}$ but not at $x = 0$, we see that $\boxed{(\pm\sqrt{3/5}, -81/125) \text{ are points of inflection}}$.
  - Since h is a polynomial of even degree with positive leading coefficient, $\lim_{x\to-\infty} h(x) = \infty = \lim_{x\to\infty} h(x)$, and there are no asymptotes.
  - Now we may assemble the graph.
    - From $x = -\infty$, h is concave up and decreasing from ∞, until it hits a local min at $(-1,-1)$ and begins increasing.
    - The function has a point of inflection at $(-\sqrt{\frac{3}{5}}, -\frac{81}{125})$, switching to concave down, and continues increasing until it hits a local max at $(0,0)$, where the function flattens out (though it doesn't change concavity).
    - Next h begins decreasing, but remains concave down to $(\sqrt{\frac{3}{5}}, -\frac{81}{125})$, where it switches to concave up.
    - Then h hits a local min at $(1,-1)$, and then finally increases (remaining concave up) out to $x = \infty$.
    - Here is the resulting graph (with the local extrema and point of inflection marked):



- <u>Example</u>: Let $f(x) = x + \frac{1}{x}$. Find the critical numbers, local minima and maxima, points of inflection, vertical and slanted asymptotes, and the intervals where f is increasing, decreasing, concave up, or concave down. Then sketch the graph of $y = f(x)$.
  - First, we analyze $f'(x) = 1 - \frac{1}{x^2}$. Note that f' is undefined at $x = 0$ (as is f itself).
    - Setting $f'(x) = 0$ yields $x^2 = 1$ so that $x = 1, -1$, so we see that $\boxed{x = -1,\ 1 \text{ are critical numbers}}$.
    - Next, we draw the number line, mark off the two critical numbers and the value $x = 0$ where f' is undefined, and then plug in test points to see that the f' sign diagram looks like $\ominus \underset{-1}{|} \oplus \underset{0}{|} \oplus \underset{1}{|} \ominus$.
    - Thus, f is $\boxed{\text{increasing on } (-1, 0) \text{ and } (0, 1)}$ and $\boxed{\text{decreasing on } (-\infty, -1) \text{ and } (1, \infty)}$.
    - Also, since f' changes from negative to positive at $x = -1$, $\boxed{(-1, -2) \text{ is a local minimum}}$, and since f' changes from positive to negative at $x = 1$, $\boxed{(1, 2) \text{ is a local minimum}}$.
  - Next, we analyze $f''(x) = \frac{2}{x^3}$.
    - Note that f'' is never zero, so there are no points of inflection.
    - To analyze concavity, we draw the number line, mark off $x = 0$ (where f'' is undefined), and then plug in two test points to see that the f'' sign diagram looks like $\ominus \underset{0}{|} \oplus$.
    - From this, we conclude that f is $\boxed{\text{concave up on } (0, \infty)}$ and $\boxed{\text{concave down on } (-\infty, 0)}$.
  - We also easily find $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$.
    - There is a vertical asymptote at $x = 0$, since we can compute $\lim_{x \to 0-} f(x) = -\infty$ and $\lim_{x \to 0+} f(x) = +\infty$.
    - Also, since $\lim_{x \to \infty} [f(x) - x] = 0$ we see that $y = x$ is a slanted asymptote as $x \to \infty$, and likewise since $\lim_{x \to -\infty} [f(x) - x] = 0$ we see that $y = x$ is also a slanted asymptote as $x \to -\infty$.
  - Now we can assemble the graph.
    - From $x = -\infty$, f is concave down and increasing until it hits a local max at $(-1, -2)$ and then decreases back down towards $-\infty$ as x approaches 0 from below.
    - There is a vertical asymptote at $x = 0$, and on the positive side of 0, f is concave up and decreasing (from ∞) until it hits a local min at $(1, 2)$, and then increases to ∞ as $x \to \infty$.
    - Furthermore, the function approaches the line $y = x$ as $x \to -\infty$ and as $x \to \infty$.
    - Here is the resulting graph (with the local extrema and asymptotes marked):



3.4 L'Hôpital's Rule

- By employing derivatives, we can evaluate certain kinds of "indeterminate" limits whose computations are not susceptible to the basic limit techniques. The most important theorem for evaluating such limits is L'Hôpital's rule.

- L'Hôpital's Rule: If $f(x)$ and $g(x)$ are differentiable functions and either $f(a) = g(a) = 0$, or both f and g diverge to ∞ or $-\infty$ as $x \to a$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, assuming the second limit exists.
  - The rule also holds for one-sided limits, or if a is $+\infty$ or $-\infty$.
  - Proof (of special case where $f(a) = g(a) = 0$ and $g'(a) \neq 0$): By the division rule for limits, and the limit definition of the derivative, we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{\left(\dfrac{f(x)-f(a)}{x-a}\right)}{\left(\dfrac{g(x)-g(a)}{x-a}\right)} = \frac{\lim_{x\to a}\left(\dfrac{f(x)-f(a)}{x-a}\right)}{\lim_{x\to a}\left(\dfrac{g(x)-g(a)}{x-a}\right)} = \frac{f'(a)}{g'(a)}.$$

  - The other cases of the rule require a more intricate and careful proof. There are several approaches; one is to use a result known as Cauchy's Mean Value Theorem.

- Important Warning: L'Hôpital's rule is not a "magic formula" for evaluating limits, and there are many indeterminate limits for which L'Hôpital's rule is not suitable. In many such cases, there are other techniques available that provide a far simpler and easier solution than L'Hôpital's rule, and in other cases L'Hôpital's rule can fail to say anything at all.

- Example $\left(\frac{0}{0}\right)$: Evaluate $\lim_{x\to 0} \frac{\tan(2x)}{x}$.
  - At $x = 0$, both $\tan(2x)$ and x take the value zero, so we can apply L'Hôpital's rule.
  - We obtain $\lim_{x\to 0} \frac{\tan(2x)}{x} = \lim_{x\to 0} \frac{2\sec^2(x)}{1} = \lim_{x\to 0} 2\sec^2(x) = 2\sec^2(0) = \boxed{2}$.

- In some cases, we may need to apply L'Hôpital's rule several times before obtaining a limit we can evaluate.

- Example $\left(\frac{0}{0}\right)$: Evaluate $\lim_{x\to 0} \frac{\sin(x)-x}{x^3}$.
  - At $x = 0$, both the numerator and denominator are zero, so we can apply L'Hôpital's rule.
  - We obtain $\lim_{x\to 0} \frac{\sin(x)-x}{x^3} = \lim_{x\to 0} \frac{\cos(x)-1}{3x^2}$. The new limit is a $\frac{0}{0}$ limit so we use L'Hôpital's rule again.
  - We get $\lim_{x\to 0} \frac{\cos(x)-1}{3x^2} = \lim_{x\to 0} \frac{-\sin(x)}{6x}$. This is still a $\frac{0}{0}$ limit, so we use L'Hôpital's rule a third time.

- We finally get $\lim_{x\to 0} \frac{-\sin(x)}{6x} = \lim_{x\to 0} \frac{-\cos(x)}{6} = -\frac{1}{6}$. Thus, the original limit $\lim_{x\to 0} \frac{\sin(x)-x}{x^3} = \boxed{-\frac{1}{6}}$.

- Example $\left(\frac{\infty}{\infty}\right)$: Evaluate $\lim_{x\to\infty} \frac{x^2}{e^x}$.
  - As x tends to infinity, e^x and x^2 both tend to $+\infty$, so we can apply L'Hôpital's rule.
  - We obtain $\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}$. The new limit is still an $\frac{\infty}{\infty}$ limit so we use L'Hôpital's rule again.
  - This yields $\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$. Thus, the original limit $\lim_{x\to\infty} \frac{x^2}{e^x} = \boxed{0}$.
  - Remark: By repeated application of L'Hôpital's rule, we can see in a similar way that $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for any value of n (even non-integral values of n): this says that e^x grows more rapidly than x^n for any value of n.

- The two limit forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ appearing in the statement of L'Hôpital's rule are called indeterminate limit forms.
  - Some other indeterminate forms are $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, and 1^∞. When these other forms occur, we will need to rearrange the given limit before L'Hôpital's rule can be applied.

- Example $(0\cdot\infty)$: Evaluate $\lim_{x\to\infty} x\cdot\left[\tan^{-1}(x) - \frac{\pi}{2}\right]$.
  - As x tends to infinity, $\tan^{-1}(x) - \frac{\pi}{2}$ tends to zero from below, so this is a $0\cdot\infty$ form.
  - We can rearrange the limit as $\lim_{x\to\infty} \frac{\tan^{-1}(x) - \frac{\pi}{2}}{1/x}$, and now it is a $\frac{0}{0}$ form.
  - Applying L'Hôpital's rule gives $\lim_{x\to\infty} \frac{\tan^{-1}(x) - \frac{\pi}{2}}{1/x} = \lim_{x\to\infty} \frac{1/(1+x^2)}{-1/(x^2)} = \lim_{x\to\infty} \frac{-x^2}{1+x^2}$.
  - Applying L'Hôpital's rule again gives $\lim_{x\to\infty} \frac{-x^2}{1+x^2} = \lim_{x\to\infty} \frac{-2x}{2x} = \lim_{x\to\infty}(-1) = -1$. (Alternatively, we could have used the polynomial trick and divided top and bottom by x^2. Using L'Hôpital's rule repeatedly will give the same result as that trick.)
  - Thus we see that $\lim_{x\to\infty} x\cdot\left[\tan^{-1}(x) - \frac{\pi}{2}\right] = \boxed{-1}$.

- Example $(\infty-\infty)$: Evaluate $\lim_{x\to(\pi/2)^-} [\sec(x) - \tan(x)]$.
  - As x tends to $\frac{\pi}{2}$ from below, both $\sec(x)$ and $\tan(x)$ blow up to $+\infty$, so this is an $\infty-\infty$ form.
  - Since $\sec(x) = \frac{1}{\cos(x)}$ and $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we can rewrite the limit as $\lim_{x\to(\pi/2)^-}\left[\frac{1-\sin(x)}{\cos(x)}\right]$, and it is now a $\frac{0}{0}$ form.
  - Applying L'Hôpital's rule gives $\lim_{x\to(\pi/2)^-}\left[\frac{1-\sin(x)}{\cos(x)}\right] = \lim_{x\to(\pi/2)^-}\left[\frac{-\cos(x)}{-\sin(x)}\right] = \frac{0}{1} = 0$.
  - So we see that $\boxed{\lim_{x\to(\pi/2)^-} [\sec(x) - \tan(x)] = 0}$.
  - Remark: In fact, the two-sided limit exists and is zero, by the same calculation.

- Example (0^0): Evaluate $\lim_{x\to 0^+} x^x$.
  - We cannot apply L'Hôpital's rule directly, because the limit is of the indeterminate form 0^0.

- However, if we take the natural log, we get $\ln\left[\lim_{x\to 0^+} x^x\right] = \lim_{x\to 0^+}[\ln(x^x)] = \lim_{x\to 0^+}[x\ln(x)]$, because the logarithm is continuous and thus we can move it through limits.
- As x tends to 0 from the positive direction, $\ln(x) \to -\infty$. We cannot apply L'Hôpital's rule to the limit as written because it is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$: instead, we have a limit of the form $0 \cdot \infty$.
- If we rewrite the limit as $\lim_{x\to 0^+} \frac{\ln(x)}{1/x}$, now we can apply L'Hôpital's rule, because now we have something of the form $\frac{\infty}{\infty}$.
- Applying the rule gives $\lim_{x\to 0^+} \frac{\ln(x)}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-(1/x^2)} = \lim_{x\to 0^+}(-x) = 0$.
- So we get $\lim_{x\to 0^+} x\ln(x) = \ln\left[\lim_{x\to 0^+} x^x\right] = 0$.
- Exponentiating gives the original limit as $\boxed{\lim_{x\to 0^+} x^x = 1}$.

- Example (1^∞): Evaluate $\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t$, where x is an arbitrary real number.
  - This limit does not seem like one to which we can apply L'Hôpital's rule: it is of the indeterminate form 1^∞.
  - However, if we take the natural logarithm, we get $\ln\left[\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t\right] = \lim_{t\to\infty}\left[\ln\left(1+\frac{x}{t}\right)^t\right] = \lim_{t\to\infty}\left[t\ln\left(1+\frac{x}{t}\right)\right]$, because the logarithm is continuous and thus we can move it through limits.
  - As $t \to \infty$, $\ln\left(1+\frac{x}{t}\right) \to 0$, so we have a limit of the form $0 \cdot \infty$. In order to apply L'Hôpital's rule, we rewrite the limit as $\lim_{t\to\infty} \frac{\ln\left(1+\frac{x}{t}\right)}{1/t}$.
  - Applying the rule gives $\lim_{t\to\infty} \frac{\ln\left(1+\frac{x}{t}\right)}{1/t} = \lim_{t\to\infty} \frac{(-x/t^2)/(1+\frac{x}{t})}{-1/t^2} = \lim_{t\to\infty} \frac{x}{1+\frac{x}{t}} = x$.
  - Finally, exponentiating gives the original limit as $\boxed{\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t = e^x}$.
  - Note: This limit is often taken as the definition of the (natural) exponential function e^x, or (by setting $x = 1$) the definition of the number e.

- Example (L'Hôpital's rule does not work): Find $\lim_{x\to\infty} \frac{e^{\ln(x)}}{x}$.
  - If we plug in, we see that this limit is of the form $\frac{\infty}{\infty}$.
  - If we try to use L'Hôpital's rule, we obtain $\lim_{x\to\infty} \frac{e^{\ln(x)}}{x} = \lim_{x\to\infty} \frac{e^{\ln(x)} \cdot \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{e^{\ln(x)}}{x}$: this is exactly the same limit we started with! (Applying the rule more times will, of course, not change this.)
  - The correct way to evaluate this limit is to observe that $e^{\ln(x)} = x$, and so the limit is just $\lim_{x\to\infty} \frac{x}{x} = \boxed{1}$.

- Example (L'Hôpital's rule makes a limit worse): Find $\lim_{x\to 0} \frac{\sin^{10}(x)}{x^{10}}$.
  - If we plug in, we see that this limit is of the form $\frac{0}{0}$.
  - Let us use L'Hôpital's rule: we obtain $\lim_{x\to 0} \frac{\sin^{10}(x)}{x^{10}} = \lim_{x\to 0} \frac{10\sin^9(x)\cdot\cos(x)}{10x^9} = \lim_{x\to 0} \frac{\sin^9(x)\cdot\cos(x)}{x^9}$.

- When we attempt to plug in here, we see the limit is still of the form $\dfrac{0}{0}$, so we try applying the rule again: we get $\displaystyle\lim_{x\to 0} \frac{\sin^9(x)\cdot\cos(x)}{x^9} = \lim_{x\to 0} \frac{9\sin^8(x)\cdot\cos^2(x) - \sin^{10} x}{9x^8}$, which is still of the form $\dfrac{0}{0}$.
- Applying the rule again yields $\displaystyle\lim_{x\to 0} \frac{9\sin^8(x)\cdot\cos^2(x) - \sin^{10} x}{9x^8} = \lim_{x\to 0} \frac{72\sin^7(x)\cos^3(x) - 28\sin^9(x)\cos(x)}{72x^7}$, which is still of the form $\dfrac{0}{0}$.
- If we try once more, we will end up with $\displaystyle\lim_{x\to 0} \frac{504\sin^6(x)\cos^4(x) - 468\sin^8(x)\cos^2(x) + 28\sin^{10}(x)}{504x^6}$, after rather more arithmetic.
- It appears that we are making very little progress at the expense of a great deal of algebra: in fact, it turns out that we would need to apply L'Hôpital's rule another 6 times before we could plug in to evaluate the limit (and the odds of making some kind of algebra or arithmetic mistake somewhere before that point is extremely high).
- A much simpler way to evaluate this limit is to use the limit laws to reduce this limit to a simpler one: explicitly, we can write $\displaystyle\lim_{x\to 0} \frac{\sin^{10}(x)}{x^{10}} = \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)^{10} = 1^{10} = \boxed{1}$, where we used the evaluation $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ (which we computed using geometry back when we found the derivative of sine, or could alternatively evaluate with a single application of L'Hôpital's rule).

3.5 Applied Optimization (Functions of One Variable)

- In many applications (particularly ones with an economic or physical flavor), we are interested in optimizing some quantity: maximizing profit, minimizing waste, etc.
- By our analysis of minimum and maximum values, in order to maximize or minimize a function of one variable, we need only find the function's critical points, and then search among the resulting function values to find the maximum or minimum we are interested in.
  - Frequently there will be physical constraints on the variables involved; e.g., a length cannot be negative. With such constraints we also need to consider the boundary behavior, and the behavior as the variable tends to $-\infty$ or ∞ (as appropriate).
  - If there are several variables involved, we will need to choose one, and use the given information to eliminate all variables but one, so that our function of interest is of one variable only, because our current methods[1] cannot accommodate optimization problems involving more than one variable.
- Most of the work in solving an applied optimization problem occurs in translating the given information into a calculus problem. Here is a general procedure for solving such problems:
  - <u>Step 1</u>: If not already given, choose variable names and translate the given information into mathematical statements. Draw a picture, if necessary.
  - <u>Step 2</u>: Identify the target function (the one to be optimized) and any constraints.
  - <u>Step 3</u>: Solve the constraints to write the target function as a function of a single variable.
  - <u>Step 4</u>: Differentiate the target function with respect to its variable, and find the critical points.
  - <u>Step 5</u>: From the critical points, identify the minimum or maximum of the target function.
- <u>Example</u>: A cylindrical jug, with a bottom and sides but no top, has radius r dm and height h dm. If the volume of the jug is to be π liters, what is the minimum possible surface area for the jug? (<u>Note</u>: $1\,\text{dm} = 10\,\text{cm}$, and $1\,\text{L} = 1\,\text{dm}^3$.)
  - The volume of the jug is $\pi r^2 h\,\text{dm}^3$, and the surface area is $\pi(r^2 + 2rh)\,\text{dm}^2$, because the base of the jug has area $\pi r^2\,\text{dm}^2$ and the lateral surface of the jug has area $2\pi rh\,\text{dm}^2$.

[1] Using multivariable calculus, one can solve such optimization problems involving multiple variables directly, without having to reduce to having a function of only one variable.

- Our target function is the surface area $SA = \pi(r^2 + 2rh)$, and we also have the constraint $\pi = V = \pi r^2 h$. Since r and h are lengths, they are positive.
- We solve the constraint for h in terms of r (since it is easier) to get $h = \frac{1}{r^2}$. Then $SA = \pi\left(r^2 + 2r \cdot \frac{1}{r^2}\right) = \pi\left(r^2 + \frac{2}{r}\right)$.
- We differentiate $SA(r)$ with respect to r: $\frac{d(SA)}{dr} = \pi \cdot \left(2 - \frac{2}{r^2}\right)$. Setting this expression equal to zero yields $2 - \frac{2}{r^2} = 0$, so that $r = 1$.
- We can see that $r = 1$ is a minimum for the surface area (since if r is large or small then $r^2 + \frac{2}{r}$ will be large; alternatively, we can check that $SA''(1)$ is positive). So the minimum occurs at $r = 1$, and the minimum surface area is $\boxed{SA(1) = 3\pi\,\mathrm{dm}^2}$.

- Example: The three vertices of a triangle are $(1, 1)$, $(5, 5)$, and $(x, 7)$. Find the value of x which minimizes the triangle's perimeter.
  - The perimeter is the sum of the three side lengths of the triangle. By the distance formula, the perimeter is $P(x) = \sqrt{(5-1)^2 + (5-1)^2} + \sqrt{(x-1)^2 + (7-1)^2} + \sqrt{(x-5)^2 + (7-5)^2}$, and this is the function we wish to minimize.
  - Expanding the squares gives the slightly simpler expression $P(x) = \sqrt{32} + \sqrt{(x-1)^2 + 36} + \sqrt{(x-5)^2 + 4}$.
  - We then compute $P'(x) = 0 + \frac{2x-2}{2\sqrt{(x-1)^2+36}} + \frac{2x-10}{\sqrt{(x-5)^2+4}}$.
  - Cancelling a factor of 2 from each fraction and rearranging gives $\frac{x-1}{\sqrt{(x-1)^2+36}} = -\frac{x-5}{\sqrt{(x-5)^2+4}}$, and squaring both sides and cross-multiplying gives $(x-1)^2\left[(x-5)^2+4\right] = (x-5)^2\left[(x-1)^2+36\right]$.
  - Expanding out and cancelling the $(x-1)^2(x-5)^2$ term from both sides gives $4(x-1)^2 = 36(x-5)^2$; taking the square root gives $2(x-1) = \pm 6(x-5)$ so that $x = 4, 7$.
  - Checking both values shows that $x = 7$ does not actually make the derivative $P'(x)$ equal to zero, as both terms in the sum are positive. So the only critical point is $x = 4$.
  - By the properties of the target function we see that the perimeter takes its minimum at $\boxed{x = 4}$, since if x is large positive or large negative, the perimeter will be large.

- Example: A flat waffle of area π square units in the shape of an isosceles triangle is to be rolled into a cone (for ice cream). What dimensions of the triangle will fit the largest volume of ice cream inside the cone?
  - Let the base of the triangle be b units, the height of the triangle be h units, the radius of the cone be r units, and the height of the cone be d units. All of these are lengths, so they are all positive.
  - The volume of the cone is $\frac{1}{3}\pi r^2 d$, which is the function we want to maximize.
  - We also see that the area of the triangle is $\frac{1}{2}bh = \pi$. Since the base of the isosceles triangle becomes the circumference of the bottom of the cone, we see $2\pi r = b$. Finally, the height of the isosceles triangle becomes the slant height of the cone, so by the Pythagorean Theorem we see that $h = \sqrt{r^2 + d^2}$.
  - We now need to eliminate all but one variable: let's write the volume of the cone in terms of the radius r. To do this we need to solve for d in terms of r.
  - We know $2\pi r = b$ so the area condition $\frac{1}{2}bh = \pi$ says $\frac{1}{2}(2\pi r)h = \pi$, or $\pi r h = \pi$, so that $h = \frac{1}{r}$.
  - Now since $h = \sqrt{r^2 + d^2}$ we have $h^2 = r^2 + d^2$ or $d^2 = \frac{1}{r^2} - r^2$, so $d = \sqrt{\frac{1}{r^2} - r^2}$.
  - Plugging in shows that $V(r) = \frac{1}{3}\pi r^2 \cdot \sqrt{\frac{1}{r^2} - r^2}$. Moving the r^2 inside the square root as $\sqrt{r^4}$ gives a slightly simpler expression $V(r) = \frac{1}{3}\pi \cdot \sqrt{r^2 - r^6}$.

- Now we can differentiate to get $\dfrac{dV}{dr} = \dfrac{1}{3}\pi \cdot \dfrac{2r - 6r^5}{2\sqrt{r^2 - r^6}}$.
- The derivative is zero when $2r - 6r^5 = 0$, or $2r(1 - 3r^4) = 0$. The only value of r between 0 and 1 which makes this true is $r = \sqrt[4]{\dfrac{1}{3}}$. So there is only one critical point which could potentially be a maximum.
- When $r \geq 1$, the function under the square root in the denominator will be zero or negative, so we ignore these values of r. Similarly, we can ignore $r \leq 0$ since r is a length.
- By the physical setup of the problem (or by the Second Derivative Test), we see that $r = \sqrt[4]{\dfrac{1}{3}}$ is the desired maximum.
- The triangle dimensions for this r are $\boxed{b = 2\pi r = 2\pi \cdot \sqrt[4]{\dfrac{1}{3}} \approx 4.77\,\text{units}}$, and $\boxed{h = \dfrac{1}{r} = \sqrt[4]{3} \approx 1.32\,\text{units}}$.
- Remark: For this optimal cone, the radius is $\sqrt[4]{\dfrac{1}{3}} \approx 0.76\,\text{units}$ and the height is $\sqrt{\sqrt{3} - \dfrac{1}{\sqrt{3}}} \approx 1.07\,\text{units}$. (Based on these dimensions, we can see that real ice cream cones are decidedly not shaped to contain the maximal amount of ice cream!)

3.6 Antiderivatives and Their Applications

- Now that we know how to differentiate functions, we might ask whether we can reverse the process: in other words, given a function f, is there a function g whose derivative is f?
  - For example, if $f(x) = 2x$ then we could take $g(x) = x^2$. Or we could take $g(x) = x^2 + 2$.
- It turns out that (provided f is continuous) the answer is yes: there is a function g whose derivative is f, although g might be much more complicated than f.
  - Such a function g can be found by integrating the function f on an appropriate interval. Since we have not discussed integration yet, we will postpone most of the discussion on how to find antiderivatives until later.
- Definition: Any function $F(x)$ whose derivative is $f(x)$ is called an antiderivative of f.
  - All of our methods for calculating derivatives immediately give us ways to find antiderivatives: we simply apply the rules "in reverse". For example, because the derivative of x^n is nx^{n-1}, this means that x^n is an antiderivative of nx^{n-1}.
  - Example: An antiderivative of $f(x) = x^3 + 4x + 2$ is $F(x) = \dfrac{1}{4}x^4 + 2x^2 + 2x$, because we can easily compute that $F'(x) = f(x)$.
  - Example: An antiderivative of $g(x) = x\cos(x^2)$ is $G(x) = \dfrac{1}{2}\sin(x^2)$, because we can compute $G'(x) = \dfrac{1}{2}\cos(x^2) \cdot 2x = g(x)$ using the Chain Rule.
- We will discuss additional techniques for computing more complicated antiderivatives once we have discussed integration. But we can start by analyzing the antiderivatives of the simplest possible function: the identically zero function.
- Theorem (Zero Derivative): If $f'(x) = 0$ for every x in an interval I, then f is constant on the interval I.
  - Note that any constant function has derivative zero, so in fact we see that the functions with zero derivative are precisely the constant functions.
  - Proof: Suppose $f'(x) = 0$ for all x in I. Choose any real numbers $a < b$ in I and apply the Mean Value Theorem to the interval $[a, b]$.
  - The theorem implies that there exists a c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

- Since $f'(x)$ is identically zero on I and c is in I, we see that $0 = \dfrac{f(b) - f(a)}{b - a}$.
- Thus, so $f(b) - f(a) = 0$, and so $f(a) = f(b)$. But now since $f(a) = f(b)$ for any two numbers a and b in the interval I, we conclude that $f(x)$ must be constant on the entire interval.

- Using this result we can show that the antiderivative of a function is "almost unique":
- Corollary: If $F_1(x)$ and $F_2(x)$ are two antiderivatives of $f(x)$ on an interval I, then there is a constant C such that $F_1(x) = F_2(x) + C$ for all x in I.
  - This means that any two antiderivatives of f differ by a constant (at least, provided f is defined on the whole interval). And clearly, if F is one antiderivative of f, then $F(x) + C$ is also an antiderivative of f.
  - Proof: Define $h(x) = F_1(x) - F_2(x)$ for x in I.
  - Taking derivatives gives $h'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$, for any x in I.
  - Therefore, $h'(x)$ is the identically zero function on I: but now by the previous theorem, we conclude that $h(x)$ is a constant function, say $h(x) = C$.
  - Then $F_1(x) = F_2(x) + C$ for all x in I, as desired.
- One application of the derivative and antiderivative is to basic physics.
  - Recall that if the position of an object is given by $x(t)$, then the velocity of that object is given by the first derivative $x'(t)$, and the acceleration is given by the second derivative $x''(t)$.
  - By Newton's second law of motion ("$F = ma$"), applying a force to an object causes it to accelerate: thus, a force will change the second derivative $x''(t)$.
  - So, if we know the forces acting on an object (or equivalently, if we know the object's acceleration), along with some starting information, then by taking antiderivatives, we can determine the object's velocity and position.
- Example (constant acceleration): An object moves with constant acceleration a along the x-axis, starting at position x_0 and velocity v_0 at time $t = 0$. Find its position $x(t)$ and velocity $v(t)$ at time t.
  - We are given that $x''(t) = v'(t) = a$ is a constant, and also $v(0) = v_0$.
  - The antiderivative of the constant function a is $at+C$, so $v(t) = at+C$. Plugging in $t = 0$ gives $v(0) = C$, so $C = v_0$, and $v(t) = at + v_0$.
  - Taking the antiderivative again gives $x(t) = \dfrac{1}{2}at^2 + v_0 t + D$, and we can solve for the constant D by setting $t = 0$: $x(0) = D$, so $D = x_0$.
  - Thus the position is given by $\boxed{x(t) = \dfrac{1}{2}at^2 + v_0 t + x_0}$ and the velocity is given by $\boxed{v(t) = at + v_0}$.
- Example: A particle accelerates from rest along a line, with $a(t) = 6t^2\ \mathrm{m/sec}$ from $t = 0$ to $t = 10\,\mathrm{sec}$, and then travels at its constant velocity for another 10 seconds. How far has it traveled from its original position after the total 20 seconds?
  - For t between 0 and $10\,\mathrm{sec}$, $a(t) = 6t^2$.
  - Taking the antiderivative and noting that $v(0) = 0$, we see that $v(t) = 6 \cdot \dfrac{t^3}{3} = 2t^3$.
  - Taking the antiderivative again gives $x(t) = 2 \cdot \dfrac{t^4}{4} = \dfrac{t^4}{2}$.
  - Therefore, after 10 seconds, the particle's velocity is $2000\,\mathrm{m/sec}$, and its position is $5000\,\mathrm{m}$ from its start.
  - Over the remaining 10 seconds, the particle moves at $2000\,\mathrm{m/sec}$, so it travels $20000\,\mathrm{m}$.
  - The total distance traveled by the particle is therefore $\boxed{25000\,\mathrm{m}}$.

- By exploiting our result about functions with zero derivative in sufficiently clever ways, we can solve certain other classes of equations involving derivatives (such equations are called differential equations, and are ubiquitous in engineering and the sciences). Here is one such result:

- Proposition (Exponential Proportionality): Suppose $y(x)$ is a differentiable function on the interval I with the property that $y'(x) = k \cdot y(x)$ for some fixed constant k: in other words, y' is proportional to y. Then there is a constant C such that $y(x) = C \cdot e^{kx}$ for all x in I.
  - Proof: Suppose $y'(x) = k \cdot y(x)$, and now consider the function $f(x) = y(x) \cdot e^{-kx}$.
  - By the product and chain rules, we have $f'(x) = y'(x) \cdot e^{-kx} + y(x) \cdot e^{-kx} \cdot (-k) = e^{-kx}[y'(x) - k\, y(x)] = 0$, by the assumption that $y'(x) = k \cdot y(x)$.
  - Therefore, f' is the identically zero function, and so f is a constant function, say, $f(x) = C$.
  - This means $y(x) \cdot e^{-kx} = C$, so that $y(x) = C \cdot e^{kx}$ as claimed.

- Example (Exponential Growth): A population of adorable kittens grows at a rate proportional to its current size. If the population at year 0 is 100 kittens and the population at year 5 is 300 kittens, what will the population be at year 18?
  - Symbolically, if $P(t)$ is the population at time t years, then we have $P'(t) = k \cdot P(t)$, where k is a proportionality constant. (Note that $k > 0$ since the population is growing.)
  - From the result above, the solution has the form $P(t) = C \cdot e^{kt}$, where C is a constant: this means that the population will exhibit exponential growth.
  - We can solve for the constants using the information given: setting $t = 0$ yields $100 = P(0) = C$, so $C = 100$, and setting $t = 5$ yields $300 = 100 \cdot e^{5k}$, so $k = \ln(3)/5 \approx 0.2197$.
  - Then the population at year 18 is $P(18) = 100 \cdot e^{18k} = 100e^{18 \cdot (\ln 3)/3} \approx \boxed{5220}$ adorable kittens (to the nearest whole number).

- Example: Newton's Law of Cooling says that the rate of change in the temperature of an object is proportional to the difference between the object's current temperature and the temperature of its environment. A baked potato is taken out of a 350°F oven at hour 0 and placed in a 25°F constant-temperature room. After 2 hours, the potato's temperature has dropped to 100°F. How much longer will it take for the potato's temperature to drop to 50°F?
  - Symbolically, if the temperature at time t hours is $T(t)$ degrees F, then we have $T'(t) = k \cdot [T(t) - E]$, where E is the temperature of the environment.
  - If we apply the result above to the function $f(t) = T(t) - E$, with $f'(t) = T'(t)$ under the assumption that E is constant, then the solution has the form $f(t) = C \cdot e^{kt}$, meaning that $T(t) = E + C \cdot e^{kt}$.
  - We can solve for the constants using the information given: first, we have $E = 25$°F.
  - Setting $t = 0$ then yields $350 = T(0) = 25 + C$, so $C = 325$. Now setting $t = 2$ yields $100 = T(2) = 25 + 325 \cdot e^{2k}$, so $e^{2k} = 75/325$ and thus $k = \frac{1}{2}\ln(75/325) \approx -0.7332$.
  - To find when the temperature will drop to 50°F we set $T(t) = 50$ and solve for t: this yields $50 = 25 + 325 \cdot e^{kt}$, so $t = \frac{1}{k}\ln(25/325) \approx 3.498$ hours. Since 2 hours have already elapsed, it will take an additional $\boxed{1.498}$ hours for the potato to cool to 50°F.

- As a final remark, we will say that in physics, chemistry, biology, engineering, and economics, virtually any interesting process is modeled by a differential equation or a system of differential equations.
  - Some other situations where differential equations naturally arise include the study of disease transmission rates, analysis of molecular interactions in chemical reactions, marginal cost and marginal profit in economics, current in electrical circuits, growth rates of organisms and populations, the mixing of compounds, movement of particles in electromagnetic or gravitational fields, and hundreds of other places.

Well, you're at the end of my handout. Hope it was helpful.