Calculus I (part 2s): Trigonometric Limits (by Evan Dummit, 2016, v. 1.00)

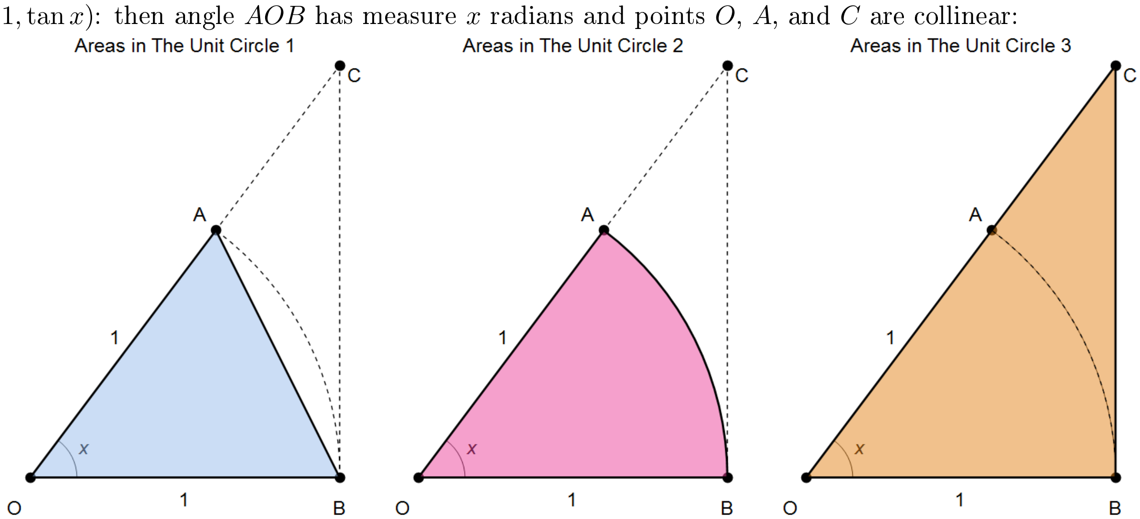
Contents

2 Introduction to Differentiation

In this supplement, we discuss trigonometric limits and some of their applications.

2.8 Trigonometric Limits

- Our first goal is to compute the limit $\lim_{x \to 0} \frac{\sin(x)}{x}$.
- Theorem (Sine Limit): The value of the limit $\lim_{x \to 0} \frac{\sin(x)}{x}$ is 1.
  - Proof: Let $0 < x < \pi/2$. Consider the unit circle, and draw points $O(0,0)$, $A(1,0)$, $B(\cos x, \sin x)$, and $C(1, \tan x)$: then angle AOB has measure x radians and points O, A, and C are collinear:



  - Now observe that triangle ABO is contained inside the circular sector ABO. The area of triangle ABO is $\frac{1}{2}\sin(x)$, since its base is 1 and its height is $\sin(x)$, while the area of sector ABO is $\frac{1}{2}x$.
  - Therefore, $\frac{1}{2}\sin(x) < \frac{1}{2}x$, or equivalently, $\frac{\sin(x)}{x} < 1$.
  - Next, observe that sector ABO is contained inside the triangle BOC. The area of triangle BOC is $\frac{1}{2}\tan(x)$, since its base is 1 and its height is $\tan(x)$, while the area of sector ABO is again $\frac{1}{2}x$.
  - Therefore, $\frac{1}{2}x < \frac{1}{2}\tan(x)$, or equivalently, $\cos(x) < \frac{\sin(x)}{x}$.
  - Combining the two inequalities, we see that $\cos(x) < \frac{\sin(x)}{x} < 1$, for all $0 < x < \frac{\pi}{2}$.
  - Since $\cos(-x) = \cos(x)$ and $\frac{\sin(-x)}{-x} = \frac{\sin(x)}{x}$, in fact the inequality $\cos(x) < \frac{\sin(x)}{x} < 1$ holds for all $-\frac{\pi}{2} < x < \frac{\pi}{2}$, except $x = 0$.

- Now because $\lim_{x\to 0} \cos(x) = 1$ and that $\lim_{x\to 0} 1 = 1$ as well, by applying the squeeze theorem we conclude that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

- Next we compute a pair of limits related to cosine: $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$ and $\lim_{x\to 0} \frac{1-\cos(x)}{x}$.
- Theorem (Cosine Limit): The value of the limit $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$ is $\frac{1}{2}$, and the value of $\lim_{x\to 0} \frac{1-\cos(x)}{x}$ is 0.
  - Proof: For the first limit, we have

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos(x)}{x^2} &= \lim_{x\to 0} \frac{1-\cos(x)}{x^2} \cdot \frac{1+\cos(x)}{1+\cos(x)} = \lim_{x\to 0} \frac{1-\cos^2(x)}{x^2} \cdot \frac{1}{1+\cos(x)} \\ &= \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1+\cos(x)} = \left[\lim_{x\to 0} \frac{\sin(x)}{x}\right]^2 \cdot \left[\lim_{x\to 0} \frac{1}{1+\cos(x)}\right] = 1^2 \cdot \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

    For the second limit, $\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \frac{1-\cos(x)}{x^2} \cdot x = \frac{1}{2} \cdot 0 = 0$.

- By manipulating these limits in sufficiently clever ways, we can compute a number of others.
- Example: Find $\lim_{t\to 0} \frac{\sin(4t)}{t}$.
  - In the sine limit $\lim_{x\to 0} \frac{\sin(x)}{x}$, if we set $x = 4t$, then $x \to 0$ is the same as saying that $t \to 0$, so upon making the change of variables, we see that $\lim_{t\to 0} \frac{\sin(4t)}{4t} = 1$.
  - Multiplying through by 4 then yields the desired $\lim_{t\to 0} \frac{\sin(4t)}{t} = \boxed{4}$.
- Example: Find $\lim_{t\to 0} \frac{\sin(3t)}{\sin(2t)}$.
  - In the sine limit $\lim_{x\to 0} \frac{\sin(x)}{x}$, if we set $x = 3t$, then as in the example above we see that $\lim_{t\to 0} \frac{\sin(3t)}{3t} = 1$.
  - Similarly, if instead we set $x = 2t$, we see that $\lim_{t\to 0} \frac{\sin(2t)}{2t} = 1$, so that $\lim_{t\to 0} \frac{2t}{\sin(2t)} = 1$ as well.
  - We can then write the original limit as $\lim_{t\to 0} \frac{\sin(3t)}{\sin(2t)} = \lim_{t\to 0} \frac{\sin(3t)}{3t} \cdot \frac{3t}{2t} \cdot \frac{2t}{\sin(2t)} = 1 \cdot \frac{3}{2} \cdot 1 = \boxed{\frac{3}{2}}$, using our evaluations above.
- Example: Find $\lim_{t\to 0} \frac{1-\cos(5t)}{\sin^2(3t)}$.
  - As in the examples above we see that $\lim_{t\to 0} \frac{\sin(3t)}{3t} = 1$, so that $\lim_{t\to 0} \frac{3t}{\sin(3t)} = 1$.
  - Also, by setting $x = 5t$ in the limit $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$, we see that $\lim_{t\to 0} \frac{1-\cos(5t)}{(5t)^2} = 1$.
  - We can then write $\lim_{t\to 0} \frac{1-\cos(5t)}{\sin^2(3t)} = \lim_{t\to 0} \frac{1-\cos(5t)}{(5t)^2} \cdot \frac{(5t)^2}{(3t)^2} \cdot \left(\frac{3t}{\sin(3t)}\right)^2 = 1 \cdot \frac{5^2}{3^2} \cdot 1 = \boxed{\frac{25}{9}}$.

Well, you're at the end of my handout. Hope it was helpful.