Calculus I (part 2): Introduction to Differentiation (by Evan Dummit, 2016, v. 2.55)

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2 Introduction to Differentiation

In this chapter, we develop the derivative along with techniques for differentiation. We begin by defining the derivative as a limit and then use the definition to compute derivatives of simple functions. We then prove the basic rules (such as the Product Rule and the Chain Rule) that will allow us to compute more complicated derivatives, and mention additional time-saving techniques such as logarithmic differentiation. We then discuss implicit relations and implicit functions (and how to compute their derivatives), and problems involving related rates and the idea of the linearization of a function.

2.1 Motivation: Rates and Tangent Lines

- We are often interested in the rate at which processes occur, both in the physical sciences and the world in general.
  - For example, we might want to know how fast a ball falls, how fast a chemical reaction occurs, how fast a population is growing.
  - Or we might be interested in how fast our car is going, how fast money is accruing in our bank accounts, and how fast we are learning something.

- We would like a way to quantify this idea precisely.

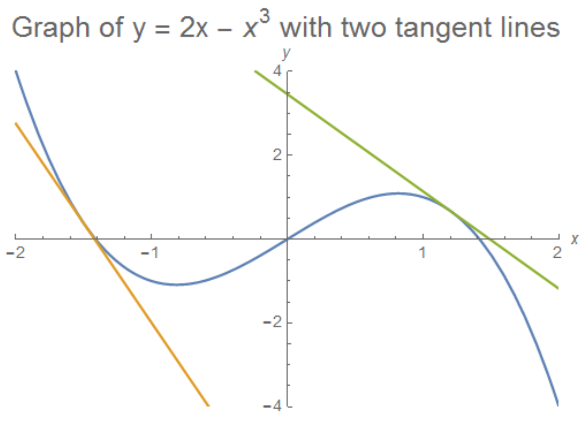
- The <u>derivative</u> of a function captures how fast that function is changing, at the point we measure it. In other words, the derivative measures the "rate of change" of a function.
  - Think of looking at a car's speedometer: it measures how fast the car is moving at the specific instant the speedometer is read.

- One notion we already can talk about is "average velocity": [average velocity]=[total distance] / [total time].
  - <u>Example</u>: A world-class sprinter can run the 100m dash in 10 seconds. Over that time, the sprinter is traveling at an average speed of 10 meters per second.

- However, we are not asking for an *average* rate of change; we want to know the *instantaneous* rate of change, which is the velocity at a specific instant, rather than over a time interval.

- One thing we might try is to calculate average rates of change over smaller and smaller intervals around our point of interest, to see if these numbers approach some value.
  - If the position function is given by $f(t)$, and we want to look over the time interval $[t, t+\Delta t]$ where Δt is some "very small" time increment, then the average velocity over that time interval is [average velocity] = $\dfrac{[\text{end position}] - [\text{start position}]}{[\text{total time}]} = \dfrac{f(t+\Delta t) - f(t)}{\Delta t}$.
  - We would like to examine this "difference quotient" as we make the time interval Δt smaller and smaller. We can do this precisely using limits, and this leads us to the formal definition of the derivative.

- Another important use of the the derivative is in finding the equation for the line tangent to a curve at a point.
  - If $f(x)$ is a "nice" function, then the <u>tangent line</u> to the graph $y = f(x)$ at the point $(a, f(a))$ is the line passing through $(a, f(a))$ that just touches the graph at $(a, f(a))$, and, near that point of tangency, is the line which is "closest" to the graph of $y = f(x)$.
  - Here is a graph of $y = 2x - x^3$ along with two tangent lines to the curve:

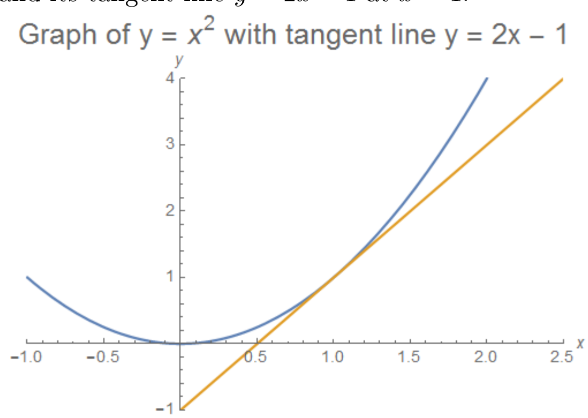


  - To specify the equation of a line requires two pieces of information: the slope of the line and one point that the line passes through.
  - The tangent line to $y = f(x)$ at $x = a$, by definition, passes through the point $(a, f(a))$, so in order to determine the tangent line, it is enough to find the slope of the tangent line.
  - One way we can try to compute the slope of the tangent line is by approximating the tangent line with "secant lines" passing through two points of the graph of $y = f(x)$: through $(a, f(a))$ and another nearby point $(a+h, f(a+h))$, where h is very small.
  - The slope of the secant line through $(a, f(a))$ and $(a+h, f(a+h))$ is, by the slope formula, given by $\dfrac{f(a+h) - f(a)}{(a+h) - a} = \dfrac{f(a+h) - f(a)}{h}$.

- Then, if f behaves well enough, we should be able to find the slope of the tangent line by evaluating the slope $\frac{f(a+h)-f(a)}{h}$ of the secant line through $(a, f(a))$ and $(a+h, f(a+h))$, as h gets closer and closer to 0.
- Again, using limits, we can formalize this notion, and (once again) the answer is given by the derivative.

2.2 Formal Definition of the Derivative

- Definition: Provided the limit exists, we define the derivative of f at x as $\boxed{f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}$.
  - Alternate Definition: An alternate way of writing the definition of the derivative is $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$.
  - Note that the limit in the alternate definition is the same as the limit in the definition above, upon making the change of variables $h = x - a$.
  - A function whose derivative exists at $x = a$ is called differentiable at $x = a$. If a function is differentiable for every value of x, we say it is everywhere differentiable.
  - Most of the functions we will encounter are differentiable everywhere that they are defined, although in general, functions do not need to be differentiable anywhere.
- The derivative $f'(x)$ measures the $\boxed{\text{instantaneous rate of change}}$ of f at x.
- The derivative $f'(x)$ is also the $\boxed{\text{slope of the tangent}}$ line to the graph of $y = f(x)$.
  - These are the most fundamental interpretations of the derivative: the author of these notes calls them "the calculus mantra".
- Example: Use the definition of the derivative to compute $f'(x)$, for $f(x) = x^2$, and then find an equation for the tangent line to $y = x^2$ at $x = 1$.
  - We have $f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = \boxed{2x}$.
  - By this calculation, we know that the tangent line to $y = x^2$ at $x = 1$ has slope $f'(1) = 2$. This tangent line passes through $(1, 1)$, and thus by the point-slope formula has equation $y - 1 = 2(x - 1)$, or $y = 2x - 1$.
  - Here is a graph of $y = x^2$ and its tangent line $y = 2x - 1$ at $x = 1$:



- Non-Example: Show that the derivative of $f(x) = |x|$ does not exist at $x = 0$.
  - If $f(x) = |x|$, then if we try to compute $f'(0)$, we have to evaluate the limit $\lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$.
  - But this limit does not exist because the two one-sided limits have unequal values, -1 and $+1$.

- Definition: The second derivative of f, denoted $f''(x)$ or $\frac{d^2 f}{dx^2}$, is the derivative of the derivative of f.
    - In general we mostly care about the first and second derivatives of $f(x)$, as they have the most meaning in physical applications.
    - But there is no reason to stop with just the second derivative: in general, we define the nth derivative of f, denoted $f^{(n)}(x)$ or $\frac{d^n f}{dx^n}$, to be the result of taking the derivative of f, n times.
- Example: Find the second and third derivatives of $f(x) = x^2$.
    - We already computed from the definition that $f'(x) = 2x$.
    - So to compute the second derivative of f, we just need to find the derivative of $2x$.
    - We have $\frac{d}{dx}[2x] = \lim_{h\to 0} \frac{2(x+h) - 2x}{h} = \lim_{h\to 0} \frac{2x + 2h - 2x}{h} = \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2$.
    - So for $f(x) = x^2$, we see that $\boxed{f''(x) = 2}$ (for every value of x).
    - For the third derivative, we need to find the derivative of $f''(x) = 2$: we obtain

$$\frac{d}{dx}[2] = \lim_{h\to 0} \frac{2-2}{h} = \lim_{h\to 0} 0 = 0$$

    and so $f'''(x) = \boxed{0}$ (for every value of x).
- In the particular event that a function represents the position $p(t)$ of an object moving along a one-dimensional axis, with t representing time, the first and second derivatives have particularly concrete interpretations:
- Definition: If an object has position $p(t)$ at time t, the first derivative $p'(t)$ represents the velocity of the object, and the second derivative $p''(t)$ represents the acceleration of the object.
    - Another way of phrasing these definitions is that velocity is the rate of change of position, and acceleration is the rate of change of velocity.
    - If the units of $p(t)$ are distance, then the units of $p'(t)$ are distance per time, and the units of $p''(t)$ are distance per time squared.
    - For example, if $p(t)$ is measured in meters and t is measured in seconds, then the units of $p'(t)$ are m/s (meters per second) and the units of $p''(t)$ are $\mathrm{m/s^2}$ (meters per second squared, or equivalently meters per second per second).
    - These units come directly out of the limit definition, because in the difference quotient $p'(t) = \lim_{h\to 0} \frac{p(t+h) - p(t)}{h}$, the numerator has units of distance while the denominator has units of time. Likewise, in $p''(t) = \lim_{h\to 0} \frac{p'(t+h) - p(t)}{h}$, the numerator has units of distance per time, while the denominator still has units of time.
    - Remark: Derivatives past the second derivative carry less obvious physical meaning, but the third derivative $p'''(t)$, which represents the rate of change of acceleration, is sometimes called jerk.
- Remarks on Notation: We will frequently write both $\frac{df}{dx}$ and $f'(x)$ to denote the derivative of f with respect to x. They mean the same thing, but we will typically use $f'(x)$ since it is easier to write, and use $\frac{df}{dx}$ only when we need to emphasize the "difference quotient" nature of the derivative.
    - Another notation we will use for the derivative is $\frac{d}{dx}[f]$: the symbol $\frac{d}{d\star}$ means "take the derivative of the thing that follows, with respect to $\star$". We will generally use this notation when f is something complicated, to make it clearer what we are doing.
    - When f is a function of a single variable, the notation f' always means the derivative of f with respect to that variable. Taking additional derivatives is denoted by adding primes: thus $f'''(x)$ is the third derivative of f.

- For functions involving more than one variable, the notation f' is ambiguous, and should not be used. For such functions, the notation f_x is often used (in place of f') to denote the derivative of f with respect to x.
- On occasion, especially in physics, the notation $\dot{f}$ is used to denote the derivative of f with respect to a variable representing time (t), in contrast with the notation f' used to denote the derivative of f with respect to a variable representing space (x).

- Roughly speaking, a differentiable function (one whose derivative exists everywhere) behaves nicely, at least in comparison to arbitrary functions. We have another notion of "niceness" (namely, continuity), and it turns out that being differentiable is a stronger condition than being continuous:

- Theorem (Differentiable Implies Continuous): If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.
  - This theorem says a differentiable function is continuous. The converse of this theorem is *false*: there exist functions which are continuous at a point but not differentiable there, such as $|x|$ at $x = 0$.
  - Even more pathologically, there exist functions which are continuous *everywhere* but differentiable *nowhere*.
  - Proof: By the multiplication rule for limits, we can write

$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a) = \left[\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right]\cdot\left[\lim_{x\to a}(x-a)\right] = f'(a)\cdot 0 = 0$$

    where in the last step we used the alternate definition of the derivative $f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$.
  - Therefore, $\lim_{x\to a}[f(x)-f(a)] = 0$. Now since $f(a)$ is a constant, we can pull it out of the limit and then rearrange the equality to read $\lim_{x\to a} f(x) = f(a)$.
  - But this says precisely that $f(x)$ is continuous at $x = a$, as we wanted.

2.3 Derivatives of Basic Functions

- These basic derivatives should be learned by heart:

$$\begin{aligned}
\frac{d}{dx}[c] &= 0\\
\frac{d}{dx}[x^n] &= n\,x^{n-1}\\
\frac{d}{dx}[e^x] &= e^x\\
\frac{d}{dx}[\ln(x)] &= \frac{1}{x}\\
\frac{d}{dx}[\sin(x)] &= \cos(x)\\
\frac{d}{dx}[\cos(x)] &= -\sin(x)
\end{aligned}$$

  - Note 1: The formulas for the derivatives of sine and cosine require that the angle be measured in radians, not degrees. This is the reason that we measure angles in radians: they are the most natural unit when doing calculus.
  - Note 2: In a similar way, the formula for the derivative of e^x is surprisingly simple, while (as we will see) the derivative of the more general exponential function $f(x) = a^x$ is $f'(x) = a^x \ln(a)$, which has an unpleasant natural logarithm constant factor in it. This is the reason we usually prefer to use the "natural exponential" e^x in calculus, since its derivative is the simplest. Likewise, among all the different logarithms, the one having the simplest derivative is the natural logarithm.

- Here is the proof of each of these rules, from the definition of the derivative:

- The derivative of a constant is zero, since if $f(x) = c$ for all x, then $f'(x) = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0$.
- If n is a positive integer, then for $f(x) = x^n$, we have (by the alternate definition of the derivative)

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{x^n - a^n}{x-a} \\ &= \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + x\,a^{n-2} + a^{n-1})}{x-a} \\ &= \lim_{x\to a} \left(x^{n-1} + x^{n-2}a + \cdots + x\,a^{n-2} + a^{n-1}\right) \\ &= a^{n-1} + a^{n-1} + \overset{n \text{ terms}}{\cdots} + a^{n-1} \\ &= n\,a^{n-1} \end{aligned}$$

  For other n (negative integers, rational numbers, general real numbers) the proof uses the same ideas, but is not much more enlightening.
- For $f(x) = e^x$, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} \\ &= e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} \\ &= e^x \end{aligned}$$

  because $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$, either by a calculation or by definition of the exponential.
- For $f(x) = \ln(x)$, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \ln\left(1 + \frac{h}{x}\right) = \lim_{h\to 0} \frac{x/h}{x} \ln\left(1 + \frac{h}{x}\right) \\ &= \lim_{y\to\infty} \frac{1}{x} \left[y \ln\left(1 + \frac{1}{y}\right)\right] = \frac{1}{x} \lim_{y\to\infty} y \ln\left(1 + \frac{1}{y}\right) \\ &= \frac{1}{x} \ln\left[\lim_{y\to\infty} \left(1 + \frac{1}{y}\right)^y\right] = \frac{1}{x}\ln(e) = \frac{1}{x} \end{aligned}$$

  where we made the change of variables $y = x/h$ and used the fact that $\lim_{y\to\infty} \left(1 + \frac{1}{y}\right)^y = e$, either by a calculation or by definition of the number e.
- For $f(x) = \sin(x)$, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \lim_{h\to 0} \left(\sin(x) \cdot \frac{\cos(h) - 1}{h} + \cos(x) \cdot \frac{\sin(h)}{h}\right) \\ &= \sin(x) \cdot \left[\lim_{h\to 0} \frac{\cos(h) - 1}{h}\right] + \cos(x) \cdot \left[\lim_{h\to 0} \frac{\sin(h)}{h}\right] \\ &= \cos(x) \end{aligned}$$

  because $\lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0$ and $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$.

  * To obtain the sine limit, a geometry calculation shows that $\cos(x) < \dfrac{\sin(x)}{x} < 1$ for small positive x. Then apply the squeeze theorem as $x \to 0$.
  * For the cosine limit, observe that $\dfrac{\cos(x)-1}{x} = \dfrac{\cos(x)-1}{x} \cdot \dfrac{\cos(x)+1}{\cos(x)+1} = \dfrac{-\sin^2(x)}{x \cdot (\cos(x)+1)} = -\dfrac{\sin(x)}{x} \cdot \dfrac{\sin(x)}{\cos(x)+1}$, and then note that as $x \to 0$ the first term goes to -1 while the second goes to 0.
- For $f(x) = \cos(x)$, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h} \\ &= \lim_{h\to 0} \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h} \\ &= \lim_{h\to 0} \left(\cos(x) \cdot \frac{\cos(h)-1}{h} - \sin(x) \cdot \frac{\sin(h)}{h}\right) \\ &= \cos(x) \cdot \left[\lim_{h\to 0} \frac{\cos(h)-1}{h}\right] - \sin(x) \cdot \left[\lim_{h\to 0} \frac{\sin(h)}{h}\right] \\ &= -\sin(x) \end{aligned}$$

again because $\displaystyle\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$ and $\displaystyle\lim_{h\to 0} \frac{\sin(h)}{h} = 1$.

- The derivatives of the other trigonometric functions, and the inverse trigonometric functions, often appear as well. The most important are $\tan(x)$, $\sin^{-1}(x)$, and $\tan^{-1}(x)$.

$$\begin{aligned} \frac{d}{dx}[\tan(x)] &= \sec^2(x) \\ \frac{d}{dx}[\sec(x)] &= \sec(x)\,\tan(x) \\ \frac{d}{dx}[\cot(x)] &= -\csc^2(x) \\ \frac{d}{dx}[\csc(x)] &= -\csc(x)\,\cot(x) \\ \frac{d}{dx}\left[\sin^{-1}(x)\right] &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\left[\cos^{-1}(x)\right] &= -\frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\left[\tan^{-1}(x)\right] &= \frac{1}{1+x^2} \\ \frac{d}{dx}\left[\sec^{-1}(x)\right] &= \frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx}\left[\csc^{-1}(x)\right] &= -\frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx}\left[\cot^{-1}(x)\right] &= -\frac{1}{1+x^2} \end{aligned}$$

  - The formulas for tangent, secant, cosecant, and cotangent can be obtained, with some effort, from the formal definition by reducing the calculations to sines and cosines. However, it is much easier to obtain them using some of the rules for computing derivatives (which we have not discussed yet).
  - The derivatives of the inverse trigonometric functions can be computed using the definition of derivative, but it is complicated and rather tricky; we will avoid this and instead discuss a general formula later for finding derivatives of inverse functions.

2.4 Calculating Derivatives

- In this section, we first discuss all of the basic techniques for computing derivatives, and then give a number of examples and basic applications.

2.4.1 Rules for Computing Derivatives

- In the list of rules below, f means $f(x)$, f' means $f'(x)$, and so on.
- Using these rules, we can compute the derivatives of any elementary function (namely, any function which is a combination of sums, products, quotients, exponentials, logs, and trigonometric and inverse trigonometric functions of x).
- Sum/Difference Rule: For any differentiable f and g, we have $\boxed{\dfrac{d}{dx}[f+g] = f' + g'}$ and $\boxed{\dfrac{d}{dx}[f-g] = f' - g'}$.
  - What this means is: if we have a sum (or difference) of functions, we can compute the derivative of the sum (or difference) just by differentiating each term one at a time, and then adding (or subtracting) the results.
  - Proof: We have

$$\begin{aligned} \frac{d}{dx}[f+g] &= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)] - [f(x)+g(x)]}{h} \\ &= \lim_{h\to 0} \frac{[f(x+h)-f(x)] + [g(x+h)-g(x)]}{h} \\ &= \lim_{h\to 0} \frac{[f(x+h)-f(x)]}{h} + \lim_{h\to 0} \frac{[g(x+h)-g(x)]}{h} \\ &= f'(x) + g'(x) \end{aligned}$$

    The proof for $f - g$ is the same, except with a minus sign.

- It might seem, based on the rules for sums and differences, that the derivative of a product of two functions would be the product of their derivatives. However, this is not true!
  - For example, $x^5 = x^3 \cdot x^2$, but the derivative of x^5 is $5x^4$ while the product of the derivatives of x^3 and x^2 is $3x^2 \cdot 2x = 6x^3$.
  - The correct rule turns out to be slightly more complicated.
- Product Rule: For any differentiable functions f and g, we have $\boxed{\dfrac{d}{dx}[f \cdot g] = f' \cdot g + f \cdot g'}$.
  - In the case where g is the constant function c, we obtain the Constant Multiple Rule $\boxed{\dfrac{d}{dx}[c\,f] = c\,f'}$.
  - Proof: We have

$$\begin{aligned} \frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{[f(x+h)g(x+h) - f(x+h)g(x)] + [f(x+h)g(x) - f(x)g(x)]}{h} \\ &= \left[\lim_{h\to 0} f(x+h)\right] \cdot \left[\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right] + \left[\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right] \cdot \left[\lim_{h\to 0} g(x)\right] \\ &= f(x)g'(x) + f'(x)g(x) \end{aligned}$$

    where in the last step, we used the fact that $\lim_{h\to 0} f(x+h) = f(x)$ because $f(x)$ is differentiable and therefore continuous.

- There is also a rule for computing derivatives of quotients. Like with the Product Rule, the derivative of a quotient is not the corresponding quotient of derivatives, but rather something slightly more complicated.

- Quotient Rule: For any differentiable functions f and g with $g \neq 0$, we have $\boxed{\dfrac{d}{dx}\left[\dfrac{f}{g}\right] = \dfrac{f' \cdot g - f \cdot g'}{g^2}}$.
  - In the Quotient Rule, it is very important not to mix up the order of the two terms in the numerator ($f'g$ has a positive sign, while $g'f$ has a negative sign). Getting them backwards will yield a result that is off by a factor of -1.
  - Proof: First we show that $\dfrac{d}{dx}\left[\dfrac{1}{g}\right] = -\dfrac{g'}{g^2}$:

$$\begin{aligned}
\frac{d}{dx}\left[\frac{1}{g(x)}\right] &= \lim_{h\to 0} \frac{\dfrac{1}{g(x+h)} - \dfrac{1}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{\left(\dfrac{g(x) - g(x+h)}{g(x+h)\cdot g(x)}\right)}{h} \\
&= \lim_{h\to 0} \frac{g(x) - g(x+h)}{h} \cdot \frac{1}{g(x)\cdot g(x+h)} \\
&= \left[\lim_{h\to 0} \frac{g(x) - g(x+h)}{h}\right] \cdot \left[\lim_{h\to 0} \frac{1}{g(x)\cdot g(x+h)}\right] \\
&= -g'(x) \cdot \frac{1}{g(x)^2}
\end{aligned}$$

    where in the last step, we used the definition of the derivative and fact that $\lim_{h\to 0} g(x+h) = g(x)$ because $g(x)$ is differentiable and therefore continuous.
  - The general result for $\dfrac{f}{g}$ follows by writing $\dfrac{f}{g} = f \cdot \dfrac{1}{g}$ and applying the Product Rule:

$$\begin{aligned}
\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{d}{dx}\left[f \cdot \frac{1}{g}\right] &= \frac{d}{dx}[f] \cdot \frac{1}{g} + f \cdot \frac{d}{dx}\left[\frac{1}{g}\right] \\
&= f' \cdot \frac{1}{g} + f \cdot \frac{-g'}{g^2} \\
&= \frac{f' \cdot g}{g^2} - \frac{f \cdot g'}{g^2} = \frac{f' \cdot g - f' \cdot g}{g^2}.
\end{aligned}$$

- The rules above allow us to differentiate any combination of sums, differences, products, and quotients.
  - However, it is not possible to describe a function like $h(x) = \sin(\cos(x))$ using only those operations and functions whose derivatives we know.
  - The missing ingredient is function composition: notice that $h(x) = f(g(x))$ where $f(x) = \sin(x)$ and $g(x) = \cos(x)$.
  - We will now give a rule for finding the derivative of a composition "chain" like $h(x) = f(g(x))$.

- Chain Rule: For any differentiable functions f and g, we have $\boxed{\dfrac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)}$.
  - Another way of writing the Chain Rule is $\boxed{\dfrac{dz}{dx} = \dfrac{dz}{dy} \cdot \dfrac{dy}{dx}}$, where z is a function of y and y is a function of x. The translation between this formulation and the explicit one above is $y = g(x)$ and $z = f(y) = f(g(x))$.
  - A natural way to interpret this formulation of the Chain Rule is using rates of change: for example, if z changes twice as fast as y and y changes four times as fast as x, then z is changing $4 \cdot 2 = 8$ times as fast as x.

- Proof[1]: By the alternate definition of the derivative,

$$\begin{aligned}\frac{d}{dx}[f(g(a))] &= \lim_{x\to a}\frac{f(g(x))-f(g(a))}{x-a} \\ &= \lim_{x\to a}\frac{f(g(x))-f(g(a))}{g(x)-g(a)}\cdot\frac{g(x)-g(a)}{x-a} \\ &= \left[\lim_{x\to a}\frac{f(g(x))-f(g(a))}{g(x)-g(a)}\right]\cdot\left[\lim_{x\to a}\frac{g(x)-g(a)}{x-a}\right] \\ &= \left[\lim_{y\to b}\frac{f(y)-f(b)}{y-b}\right]\cdot\left[\lim_{x\to a}\frac{g(x)-g(a)}{x-a}\right] \\ &= f'(b)\cdot g'(a) \\ &= f'(g(a))\cdot g'(a)\end{aligned}$$

  where to get from the third equation to the fourth, we substituted $y = g(x)$ and $b = g(a)$ in the first limit and used the fact that $y \to b$ as $x \to a$ because $g(x)$ is differentiable and therefore continuous.

- There are a few useful corollaries of the Chain Rule which are worth remembering on their own:
  - When $f(x)$ is the natural logarithm, we obtain the Log Rule $\boxed{\boxed{\frac{d}{dx}[\ln(g)] = \frac{g'}{g}}}$.
  - When $f(x)$ is the base-e exponential we obtain the Exponential Rule $\boxed{\boxed{\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}\cdot g'(x)}}$.
  - In particular, because $a^x = e^{x\ln(a)}$, if we set $g(x) = x\ln(a)$ in the rule above, we obtain the general-base exponential derivative $\boxed{\boxed{\frac{d}{dx}[a^x] = a^x\ln(a)}}$.
- Finally, we will give a rule for computing the derivative of an inverse function:
- Inverse Rule: If f is a differentiable function with inverse f^{-1}, then $\boxed{\boxed{\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}}}$.
  - Proof: By the alternate definition of derivative, we have $\frac{d}{dx}\left[f^{-1}(x)\right](a) = \lim_{x\to a}\frac{f^{-1}(x)-f^{-1}(a)}{x-a}$.
  - Now set $b = f^{-1}(a)$ and make the substitution $y = f^{-1}(x)$, so that $a = f(b)$ and $x = f(y)$.
  - We obtain $\frac{d}{dx}\left[f^{-1}(x)\right](a) = \lim_{y\to b}\frac{y-b}{f(y)-f(b)} = \frac{1}{\lim_{y\to b}\frac{f(y)-f(b)}{y-b}} = \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}$, as desired.

2.4.2 Basic Examples of Derivatives

- The best way to become comfortable computing derivatives is to work examples. Here are some simpler ones:
- Example: Differentiate $f(x) = \sqrt{x}$ with respect to x.
  - Note that $\sqrt{x} = x^{1/2}$, so applying the Power Rule yields $f'(x) = \boxed{\frac{1}{2}x^{-1/2}}$.
- Example: Differentiate $g(x) = 3x^{10} + 3x^{-1} - x^{1/2}$ with respect to x.

[1] This proof has a minor issue, which is that $g(x) - g(a)$ might be zero even for x very close to a; an example occurs for $a = 0$ with the function $g(x) = x^3\cos(1/x)$. To fix this issue one must analyze the function $\frac{f(g(x))-f(g(a))}{g(x)-g(a)}$ more carefully; the precise details are not especially enlightening and so we omit them.

- Here, we can just differentiate term-by-term (secretly applying the Sum and Difference Rules) using the Power Rule to get $\boxed{g'(x) = 30x^9 - 3x^{-2} - \frac{1}{2}x^{-1/2}}$.

- Example: For $h(y) = e^y \cdot \ln(y)$, find $h'(y)$.
  - The Product Rule gives $h'(y) = e^y \cdot \frac{d}{dy}[\ln(y)] + \frac{d}{dy}[e^y] \cdot \ln(y) = \boxed{e^y \cdot \frac{1}{y} + e^y \cdot \ln(y)}$.

- Example: Differentiate $q(t) = \dfrac{e^t + 3}{t^2}$ with respect to t.
  - The Quotient Rule gives $q'(t) = \dfrac{\frac{d}{dt}[e^t + 3] \cdot (t^2) - \frac{d}{dt}[t^2] \cdot (e^t + 3)}{(t^2)^2} = \boxed{\dfrac{e^t \cdot (t^2) - 2t \cdot (e^t + 3)}{(t^2)^2}}$.
  - This can be simplified by cancelling a factor of t, giving $\boxed{q'(t) = \dfrac{t\,e^t - 2(e^t + 3)}{t^3}}$.

- Example: Differentiate $f(x) = \tan(x)$ with respect to x.
  - If we write $f(x) = \sin(x)/\cos(x)$, we can apply the Quotient Rule to see

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}[\sin(x)] \cdot (\cos(x)) - \frac{d}{dx}[\cos(x)] \cdot (\sin(x))}{\cos^2(x)} \\ &= \frac{\cos(x) \cdot \cos(x) - (-\sin(x)) \cdot \sin(x)}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} \\ &= \boxed{\sec^2(x)}. \end{aligned}$$

  - The derivatives of secant, cosecant, and cotangent can be computed the same way.

- Example: Find the derivative of $h(x) = (x^2 + e^x)^{15}$.
  - Here, we can apply the Chain Rule, with $f(x) = x^{15}$ and $g(x) = x^2 + e^x$.
  - We obtain $h'(x) = f'(g(x)) \cdot g'(x) = \boxed{15(x^2 + e^x)^{14} \cdot (2x + e^x)}$.

- Example: Differentiate $k(x) = \sqrt{x^3 + 1}$.
  - Here, we can apply the Chain Rule, with $f(x) = \sqrt{x} = x^{1/2}$ and $g(x) = x^3 + 1$.
  - We obtain $k'(x) = f'(g(x)) \cdot g'(x) = \boxed{\frac{1}{2}(x^3 + 1)^{-1/2} \cdot 3x^2}$.

- Example: For $g(x) = \tan^{-1}(x)$, find $g'(x)$.
  - We use the Inverse Rule: notice that $g(x) = f^{-1}(x)$ for $f(x) = \tan(x)$.
  - Since $f'(x) = \sec^2(x)$, we get $g'(x) = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{1 + \tan^2(\tan^{-1}(x))} = \boxed{\dfrac{1}{1 + x^2}}$.
  - The derivatives of the other inverse trigonometric functions can be computed the same way.

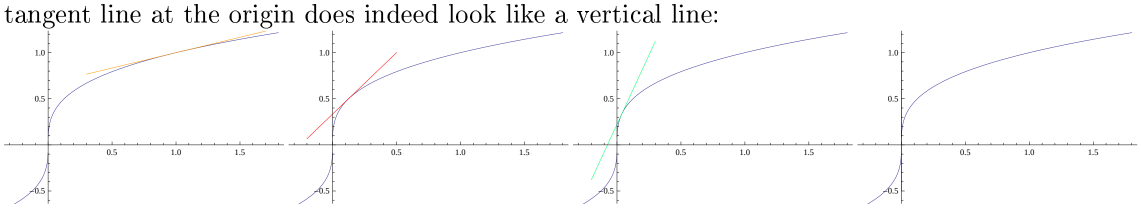
- Using these differentiation techniques, we can quickly solve the kinds of problems that motivated our definition of the derivative without having to resort to cumbersome limit calculations:

- Example: Find an equation for the tangent line to $y = x^2 + \dfrac{3}{x}$ at $x = 1$.

- We have $y(1) = 4$, so the tangent line passes through $(1, 4)$.
- The derivative of this function is $y' = 2x - \frac{3}{x^2}$, so $y'(1) = -1$.
- Therefore, the tangent line has slope -1 and passes through the point $(1, 4)$, so its equation is $\boxed{y - 4 = -1(x - 1)}$. (Note that we could also write the equation in a variety of other ways, such as $y = 5 - x$ or $x + y = 5$.)

- Example: If at time t seconds, a particle has position $p(t) = t - \sin(\pi t)$ meters, how fast is it moving at $t = 2$s and how fast is it accelerating at $t = 3$s?
  - The velocity is given by the derivative of position, so $v(t) = p'(t) = 1 - \pi\cos(\pi t)$, so at $t = 2$s, we see that $v(2) = 1 - \pi\cos(2\pi) = 1 - \pi$.
  - The units on the velocity are meters per second, and since $1 - \pi$ is negative and speed is positive, at time $t = 2$s we see that the particle is moving at $\boxed{(\pi - 1)\text{m/s}}$.
  - Acceleration is the derivative of velocity, and the units on acceleration are meters per second squared, m/s^2, so $a(t) = v'(t) = \pi^2 \sin(\pi t)$, so at $t = 3$s, we see that $a(3) = \boxed{0\text{m/s}^2}$. (This says that the particle is not accelerating either forwards or backwards at time $t = 3$s.)

- Example: Find an equation for the tangent line to $y = x^{1/3}$ at $x = 1$, at $x = \frac{1}{8}$, at $x = \frac{1}{27}$, and explain what happens at $x = 0$.
  - The derivative is $y' = \frac{1}{3}x^{-2/3}$.
  - We have $y(1) = 1$ so the first line passes through $(1, 1)$. Its slope is $y'(1) = \frac{1}{3}$, so its equation is $\boxed{y - 1 = \frac{1}{3}(x - 1)}$, or $y = \frac{1}{3}x + \frac{2}{3}$.
  - We have $y(\frac{1}{8}) = \frac{1}{2}$, and $y'(\frac{1}{8}) = \frac{4}{3}$, so the second line's equation is $\boxed{y - \frac{1}{2} = \frac{4}{3}(x - \frac{1}{8})}$, or $y = \frac{4}{3}x + \frac{1}{3}$.
  - We have $y(\frac{1}{27}) = \frac{1}{3}$, and $y'(\frac{1}{27}) = 3$, so the third line's equation is $\boxed{y - \frac{1}{3} = 3(x - \frac{1}{27})}$, or $y = 3x + \frac{2}{9}$.
  - At the origin $(0, 0)$, we see that the derivative y' is undefined: if we take the limit as $x \to 0$ from either side, we see that $y' \to +\infty$. (We will often abuse the terminology slightly and say that the derivative is "infinite" in a case like this.) Under the usual interpretation of "a line with slope ∞" as a vertical line, we guess that the tangent line to this curve is vertical at the origin and thus has equation $\boxed{x = 0}$.
  - Indeed, this is borne out by the graph of the curve and the three tangent lines we found above – the tangent line at the origin does indeed look like a vertical line:



2.4.3 Additional Examples of Derivatives

- The examples we have given so far involve essentially a single application of one of the differentiation rules. However, sufficiently complicated functions will often require use of several of the rules in tandem.
  - Although it may seem difficult, the procedure is very algorithmic: simply decide which which rule can be applied first. Then write down the result of applying that rule to the given expression, and then evaluate any remaining derivatives in the same way.

- Example: Differentiate $f(x) = \ln\left(1 + e^{\sqrt{x}}\right)$.

- First, observe that we can apply the Chain Rule to see that $f'(x) = \dfrac{\frac{d}{dx}(1+e^{\sqrt{x}})}{1+e^{\sqrt{x}}}$.
- Now we are left with the simpler task of evaluating the derivative in the numerator. We can do this with another application of the Chain Rule, yielding the end result $f'(x) = \boxed{\dfrac{e^{\sqrt{x}} \cdot \frac{1}{2}x^{-1/2}}{1+e^{\sqrt{x}}}}$.

- Example: Differentiate $f(x) = \dfrac{x^3 + 2\sqrt{x}}{\sin(x)}$.
  - We start by applying the Quotient Rule, giving $f'(x) = \dfrac{\frac{d}{dx}[x^3+2\sqrt{x}] \cdot \sin(x) - \frac{d}{dx}[\sin(x)] \cdot (x^3+2\sqrt{x})}{(\sin(x))^2}$.
  - Evaluating the leftover derivatives yields $f'(x) = \boxed{\dfrac{(3x^2+x^{-1/2}) \cdot \sin(x) - \cos(x) \cdot (x^3+2\sqrt{x})}{(\sin(x))^2}}$.

- Example: Differentiate $h(r) = r^3 2^r \ln(r)$.
  - Here, we have a product of 3 terms. We can apply the Product Rule to this expression by grouping two of the terms together: $h(r) = \left[r^3 2^r\right] \cdot \ln(r)$.
  - We obtain

$$\begin{aligned} h'(r) &= \frac{d}{dr}\left[r^3 2^r\right] \cdot \ln(r) + r^3 2^r \cdot \frac{d}{dr}\left[\ln(r)\right] \\ &= \boxed{\left[3r^2 2^r + r^3 2^r \ln(2)\right] \cdot \ln(r) + r^3 2^r \cdot \frac{1}{r}} \end{aligned}$$

  where we used the Product Rule again to differentiate $r^3 2^r$.

- Example: Differentiate $f(x) = \sin(\sin(\sin(\sin(x))))$.
  - This function seems intimidating, but it is built entirely through function composition, so we can find its derivative using the Chain Rule by working from the outside to the inside.
  - To start, note that $f(x)$ has the form $\sin(\star)$, where $\star$ is some other function (namely, $\sin(\sin(\sin(x)))$).
  - Therefore, $f'(x) = \cos(\star) \cdot \dfrac{d}{dx}[\star]$ by the Chain Rule.
  - We have now reduced the problem to finding the derivative of the simpler expression $\star = \sin(\sin(\sin(x)))$.
  - Repeatedly applying the Chain Rule in this manner yields

$$\begin{aligned} f'(x) &= \cos(\sin(\sin(\sin(x)))) \cdot \frac{d}{dx}\left[\sin(\sin(\sin(x)))\right] \\ &= \cos(\sin(\sin(\sin(x)))) \cdot \cos(\sin(\sin(x))) \cdot \frac{d}{dx}\left[\sin(\sin(x))\right] \\ &= \cos(\sin(\sin(\sin(x)))) \cdot \cos(\sin(\sin(x))) \cdot \cos(\sin(x)) \cdot \frac{d}{dx}\left[\sin(x)\right] \\ &= \boxed{\cos(\sin(\sin(\sin(x)))) \cdot \cos(\sin(\sin(x))) \cdot \cos(\sin(x)) \cdot \cos(x)}. \end{aligned}$$

- Example: Differentiate $f(x) = x\tan(x^3 + e^{2x^2\cos(x)})$.
  - By applying the Product Rule and Chain Rule repeatedly, we obtain

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x] \cdot \tan(x^3 + e^{2x^2\cos(x)}) + x \cdot \frac{d}{dx}\left[\tan(x^3 + e^{2x^2\cos(x)})\right] \\ &= 1 \cdot \tan(x^3 + e^{2x^2\cos(x)}) + x \cdot \sec^2(x^3 + e^{2x^2\cos(x)}) \cdot \frac{d}{dx}\left[x^3 + e^{2x^2\cos(x)}\right] \\ &= 1 \cdot \tan(x^3 + e^{2x^2\cos(x)}) + x \cdot \sec^2(x^3 + e^{2x^2\cos(x)}) \cdot \left[3x^2 + e^{2x^2\cos(x)} \cdot \frac{d}{dx}\left[2x^2\cos(x)\right]\right] \\ &= \boxed{1 \cdot \tan(x^3 + e^{2x^2\cos(x)}) + x \cdot \sec^2(x^3 + e^{2x^2\cos(x)}) \cdot \left[3x^2 + e^{2x^2\cos(x)} \cdot \left[4x\cos(x) - 2x^2\sin(x)\right]\right]}. \end{aligned}$$

- Example: Differentiate $q(t) = \sqrt{\dfrac{\tan^{-1}(3t^2)}{\sin(\sec(t))}}$.
    - By applying the Quotient Rule and Chain Rule repeatedly, we obtain

$$\begin{aligned} q'(t) &= \frac{1}{2}\left[\frac{\tan^{-1}(\ln(3t))}{\sin(\sec(t))}\right]^{-1/2} \cdot \frac{d}{dt}\left[\frac{\tan^{-1}(\ln(3t))}{\sin(\sec(t))}\right] \\ &= \frac{1}{2}\left[\frac{\tan^{-1}(\ln(3t))}{\sin(\sec(t))}\right]^{-1/2} \cdot \frac{\frac{1}{1+(3t^2)^2}\cdot\frac{d}{dt}[1+3t^2]\cdot\sin(\sec(t)) - \tan^{-1}(3t^2)\cdot\cos(\sec(t))\cdot\frac{d}{dt}[\sec(t)]}{\sin^2(\sec(t))} \\ &= \boxed{\frac{1}{2}\left[\frac{\tan^{-1}(\ln(3t))}{\sin(\sec(t))}\right]^{-1/2} \cdot \frac{\frac{1}{1+(3t^2)^2}\cdot 6t\cdot\sin(\sec(t)) - \tan^{-1}(3t^2)\cdot\cos(\sec(t))\cdot\sec(t)\tan(t)}{\sin^2(\sec(t))}}. \end{aligned}$$

2.4.4 Logarithmic Differentiation

- Evaluating derivatives of large products, quotients, and complicated powers can be achieved using the rules we have established, but the results can be very cumbersome.
- There is a useful trick to simplify such differentiation problems, called logarithmic differentiation: the basic idea of the procedure is to differentiate $\ln(f)$ rather than f itself. If $\ln(f)$ can be simplified using properties of logarithms, the resulting computations can be substantially easier. We will illustrate the idea with a few examples.
- Example: Differentiate $f(x) = (x-2)^3(x^2+1)^7$.
    - We could just use the Product and Chain Rules, giving $f'(x) = \boxed{3(x-2)^2\cdot(x^2+1)^7 + (x-2)^3\cdot 7(x+1)^6\cdot 2x}$.
    - Let us now solve the problem a different way: if we first take logarithms of both sides, we obtain $\ln(f) = \ln\left[(x-2)^3(x^2+1)^7\right] = 3\ln(x-2) + 7\ln(x^2+1)$.
    - Differentiating both sides then yields $\dfrac{f'}{f} = 3\cdot\dfrac{1}{x-2} + 7\cdot\dfrac{2x}{x^2+1}$.
    - Thus, $f' = f\cdot\left[3\cdot\dfrac{1}{x-2} + 7\cdot\dfrac{2x}{x^2+1}\right] = \boxed{(x-2)^3(x^2+1)^7\left[3\cdot\dfrac{1}{x-2} + 7\cdot\dfrac{2x}{x^2+1}\right]}$.
    - If we expand out the product and cancel denominator terms, we get $f' = 3(x-2)^3(x^2+1)^7 + 7(x-2)^3(x+1)^6\cdot 2x$, which is the same expression we obtained above. (Of course, this should not be surprising!)
- The logarithmic differentiation method saves the most time with complicated products, quotients, or powers.
- Example: Differentiate $f(x) = \dfrac{(x-3)^7(x^2+1)^8}{(x^3+2)^4}$.
    - In principle, we could use the Quotient, Product, and Chain Rules to solve this problem, but the calculations would be rather lengthy.
    - Instead, let us write $\ln(f) = \ln\left[\dfrac{(x-3)^7(x^2+1)^8}{(x^3+2)^4}\right] = 7\ln(x-3) + 8\ln(x^2+1) - 4\ln(x^3+2)$.
    - Differentiating both sides yields $\dfrac{f'}{f} = 7\cdot\dfrac{1}{x-3} + 8\cdot\dfrac{2x}{x^2+1} - 4\cdot\dfrac{3x^2}{x^3+2}$.
    - Thus, $f' = f\cdot\left[\dfrac{7}{x-3} + 8\cdot\dfrac{2x}{x^2+1} - 4\cdot\dfrac{3x^2}{x^3+2}\right] = \boxed{\dfrac{(x-3)^7(x^2+1)^8}{(x^3+2)^4}\left[\dfrac{7}{x-3} + 8\cdot\dfrac{2x}{x^2+1} - 4\cdot\dfrac{3x^2}{x^3+2}\right]}$.
- Example: Differentiate $f(x) = \dfrac{\sqrt{x^3+3}\cdot\sqrt[5]{(x^3-1)^2}}{(x^4+x+1)^{100}}$.
    - We have $\ln(f) = \dfrac{1}{2}\ln(x^3+3) + \dfrac{2}{5}\ln(x^3-1) - 100\ln(x^4+x+1)$.

- Differentiating yields $\dfrac{f'}{f} = \dfrac{1}{2} \cdot \dfrac{3x^2}{x^3+3} + \dfrac{2}{3} \cdot \dfrac{3x^2}{x^3-1} - 100 \cdot \dfrac{4x^3}{x^4+x+1}$.
- Thus, $f' = \boxed{\dfrac{\sqrt{x^3+3} \cdot \sqrt[5]{(x^3-1)^2}}{(x^4+x+1)^{100}} \cdot \left[\dfrac{1}{2} \cdot \dfrac{3x^2}{x^3+3} + \dfrac{2}{3} \cdot \dfrac{3x^2}{x^3-1} - 100 \cdot \dfrac{4x^3}{x^4+x+1}\right]}$.

- Example: Differentiate $f(x) = (x + \sin(2x))^{\tan(x)}$.
  - We write $\ln(f) = \tan(x) \cdot \ln[x + \sin(2x)]$.
  - Differentiating yields $\dfrac{f'}{f} = \sec^2(x) \cdot \ln[x + \sin(2x)] + \tan(x) \cdot \dfrac{1 + 2\cos(2x)}{x + \sin(2x)}$.
  - Thus, $f' = \boxed{(x + \sin(2x))^{\tan(x)} \cdot \left[\sec^2(x) \cdot \ln[x + \sin(2x)] + \tan(x) \cdot \dfrac{1 + 2\cos(2x)}{x + \sin(2x)}\right]}$.
- More generally, we can use logarithmic differentiation to find the derivative of a general exponential of the form $f(x)^{g(x)}$: the result is $\dfrac{d}{dx}\left[f(x)^{g(x)}\right] = \left[g'(x)\,\ln f(x) + g(x) \cdot \dfrac{f'(x)}{f(x)}\right] \cdot f(x)^{g(x)}$.
  - It is not worth memorizing this rule (which is why it is not boxed). Instead, it is much more useful to remember how to compute such derivatives using logarithmic differentiation.

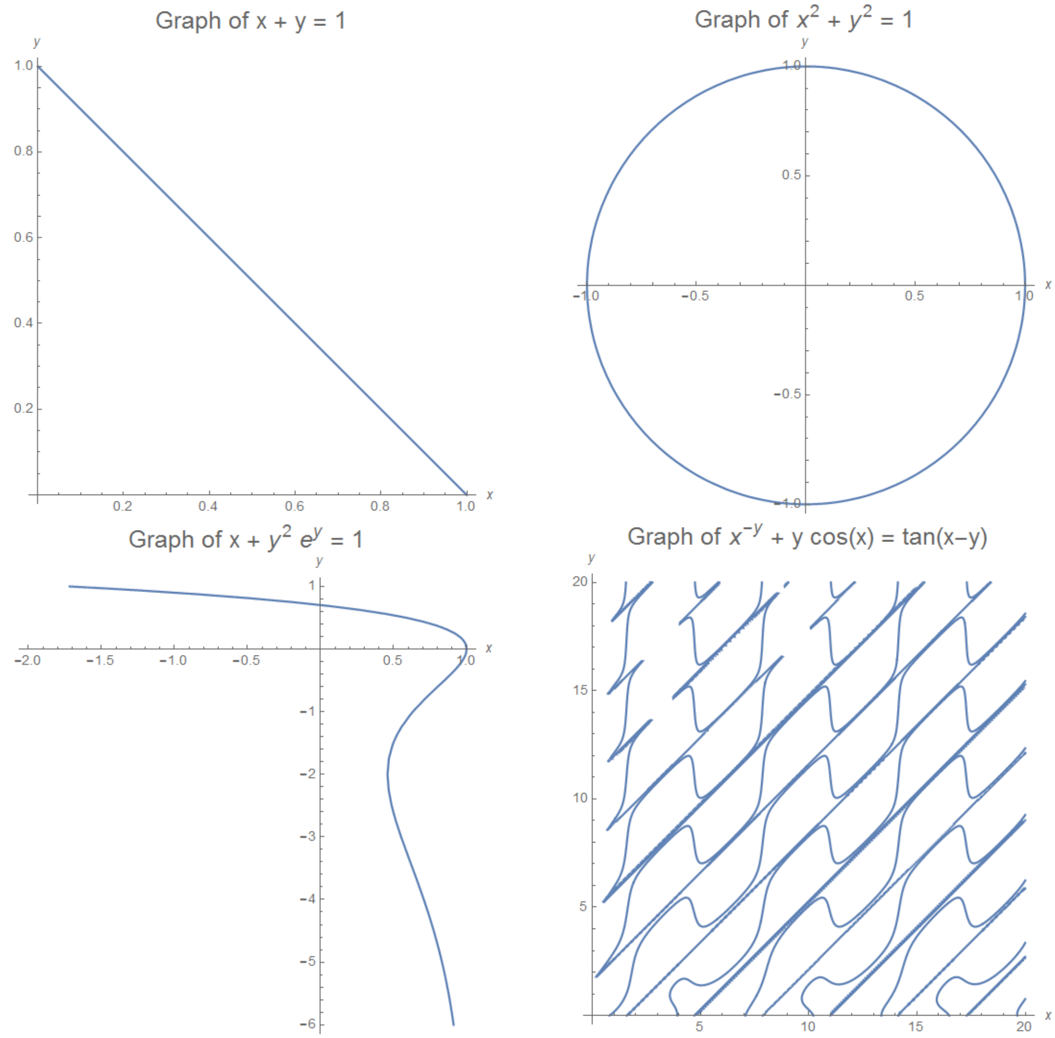
2.5 Implicit Curves and Implicit Differentiation

- With the differentiation rules, we can now compute the derivative of any function $y = f(x)$ defined explicitly as a function of x, at least as long as it involves polynomials, powers, exponentials, logarithms, or (inverse) trigonometric functions (whose derivatives we know).
- Sometimes, however, we are also interested in "implicit relations", which relate the values of y and x not via an explicit function $y = f(x)$, but as some equation of the form $R(x, y) = 0$, where R is an arbitrary function of the two variables x and y.
  - Any equation relating x and y can (by moving all the terms to one side) be written in the form $R(x, y) = 0$.
  - So "an implicit relation involving x and y" is the same thing as "some equation that x and y satisfy".

2.5.1 Implicit Relations and Implicit Functions

- Definition: An implicit relation between two variables x and y is an equation of the form $R(x, y) = 0$, where R is an arbitrary function of the two variables x and y.
  - We allow ourselves to rearrange implicit relations, to put things on either side of the equality.
  - Examples: $x + y = 1$, $x^2 + y^2 = 1$, $x + y^2\,e^y = 1$, and $x^{-y} + y\,\cos(x) = \tan(x - y)$ are four implicit relations between x and y.
- Sometimes, it is possible to turn an implicit relation into an explicit one by rearranging it. Other times, this is not possible.
  - Example: Rearranging $x + y = 1$ yields the explicit formula $y = 1 - x$ for y in terms of x.
  - Example: Rearranging $x^2 + y^2 = 1$ yields the explicit formula $y = \pm\sqrt{1 - x^2}$. Note that the plus-or-minus is necessary, because both $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ satisfy $x^2 + y^2 = 1$.
  - Example: There is no obvious way to solve $x + y^2\,e^y = 1$ to write y explicitly as a function of x. However, it is possible to write x as a function of y: $x = 1 - y^2\,e^y$.
  - Example: There is no obvious way to solve $x^{-y} + y\,\cos(x) = \tan(x - y)$ explicitly for either one of the variables in terms of the other.

- Like with explicit functions $y = f(x)$, we can produce a graph of all the points (x, y) satisfying an implicit relation.
    - Note that actually producing the graph of an implicit relation, though, is trickier than producing the graph of an explicit function $y = f(x)$: more is required than just picking a bunch of values of x and evaluating the function.
    - Usually, the graph of a single implicit relation will be a curve or a collection of curves (possibly intersecting each other) in the plane.
    - Here are graphs of $x + y = 1$, $x^2 + y^2 = 1$, $x + y^2\, e^y = 1$, and $x^{-y} + y\,\cos(x) = \tan(x - y)$, respectively:



    - As can be seen from the plot of $x^{-y} + y\,\cos(x) = \tan(x - y)$, implicit relations can have very complicated graphs!

- Motivating Example: Consider the implicit relation $x^2 + y^2 = 1$, whose graph is the unit circle.
    - To describe the graph of the unit circle explicitly we have to glue together the graphs of the two functions $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ for $-1 \le x \le 1$. But if we allow ourselves an implicit relation for y, then we can describe the unit circle with the far simpler single equation $x^2 + y^2 = 1$.
    - In the case of $x^2 + y^2 = 1$, we can solve the implicit relation for y.
    - However, for more complicated implicit relations, it may not be possible to solve for y explicitly as a function of x; for example, it is not possible to solve $y + \sin(y) = x + e^x$ for either x or y as an elementary function of the other.
    - Even if it is possible to solve for y, the result might be very complicated. And it is also possible to lose information, if we are not extremely careful about solving the relation for y.

* For example, $x = \sin(y)$ and $y = \sin^{-1}(x)$ appear to be equivalent, but in fact they aren't: $(x, y) = (0, 2\pi)$ satisfies the first equation but not the second one, because the (principal) arcsine function $\sin^{-1}$ only has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

- Remark ("relation" versus "function") : We often want to speak of implicit relations between x and y as "implicitly defining y as a function of x". Technically speaking, this is often troublesome: in any given implicit relation, there may be more than one value of y associated to a particular value of x, in which case y is not actually a function of x. If we want to insist on trying to make y an actual function of x, we would have to specify which particular value of y we want whenever there is a choice.
  - For example, consider the implicit relation $x^2 = y^2$. If we solve for y in terms of x then we obtain $y = \pm x$. For each value of x (except for $x = 0$) the relation gives two possible values of y.
  - If we wanted to solve the implicit relation $x^2 = y^2$ to obtain y as a function of x, we would need to specify which value of y we want for each possible value of x.
  - There are many ways of doing this: some of the obvious ones are $y = x$, or $y = -x$, or $y = |x|$. But we could also do something like taking $y = x$, for $-3 \le x \le 3$, and taking $y = -x$ for the other possible x.
- From here onwards, we will abuse terminology and not worry about whether the things we want to call "implicit functions" should really be named "functions" or not.

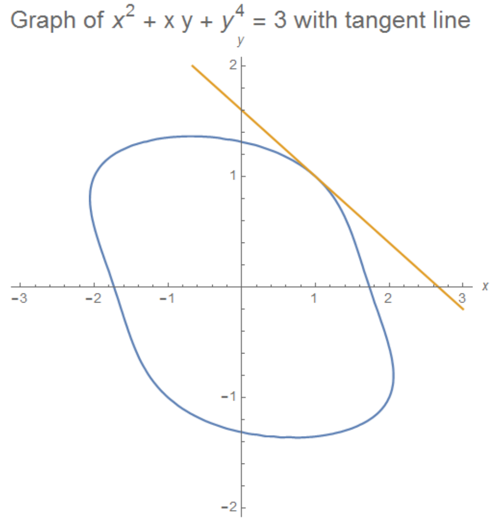
2.5.2 Implicit Differentiation

- One very natural question we might have is: if we have a point (x, y) on an implicit curve, what happens to y if we change x by a small amount?
  - If we had a curve given explicitly, changing x by a small amount Δx will cause y to change by a small amount Δy, and we will have an approximate equality $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} = y'$.
  - So the question becomes: what is the right way to think about and compute derivatives "implicitly"?
- Here is the answer: To calculate the derivative y', in a situation where y and x satisfy an implicit relation $R(x, y) = 0$, we think of y as $y(x)$, an implicit function of x, and then differentiate both sides of the relation $R(x, y) = 0$ via the Chain Rule. This will give an equation that can be solved for y' as a function of x and y.
  - Note that the interpretation of y' is still exactly the same: y' still gives the slope of the tangent line to the curve. Thus, when $y' = 0$, the tangent line is horizontal, and when y' "blows up" to $\pm\infty$, the tangent line is vertical.
  - One other type of unusual behavior can happen with implicit relations: if the graph of the curve $R(x, y) = 0$ crosses itself, then the value of y' will usually be an indeterminate fraction of the form $\frac{0}{0}$.
- Furthermore, if we want to compute the second derivative y'' in terms of x and y, then we can simply differentiate y' (using the Chain Rule), and then substitute in for any occurrences of y' in the resulting expression to write y'' as a function of x and y. Repeatedly differentiating in this manner allows calculation of any higher derivative, although the results usually get complicated very quickly.
- Example: For the implicit function y defined by $x^2 + y^2 = 1$, find y' using implicit differentiation, and then find the slope of the tangent line to its graph at the point $\left(\frac{3}{5}, -\frac{4}{5}\right)$.
  - To find y', we think of y as a function of x and differentiate both sides of $x^2 + [y(x)]^2 = 1$ with the Chain Rule.
  - This gives $2x + 2 \cdot [y(x)]^1 \cdot y'(x) = 0$, or, more compactly, $2x + 2y\, y' = 0$.
  - Solving for y' gives $2y\, y' = -2x$, so that $\boxed{y' = \frac{-2x}{2y} = -\frac{x}{y}}$.

- The slope of the tangent line at $\left(\frac{3}{5}, -\frac{4}{5}\right)$, which we can verify actually does lie on the curve since $\left(-\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1$, is then $y' = -\frac{3/5}{-4/5} = \boxed{\frac{3}{4}}$.

- Example (repeated): For the implicit function y defined by $x^2+y^2 = 1$, try to find y' by solving for y explicitly, and then find the slope of the tangent line to its graph at the point $\left(\frac{3}{5}, -\frac{4}{5}\right)$.
  - Solving $x^2 + y^2 = 1$ for y gives $y = \pm\sqrt{1-x^2} = \pm(1-x^2)^{1/2}$. Note that the $\pm$ is very much necessary because there are two possible values of the square root, positive and negative.
  - If we blindly take the derivative of the expression $y = \pm(1-x^2)^{1/2}$ with the Chain Rule, we obtain $y' = \pm\frac{1}{2} \cdot (1-x^2)^{-1/2} \cdot (-2x)$.
  - Now there is already an issue: which sign do we want? The only plausible answer is that it must depend on the value of y we are interested in.
  - For $y = -4/5$, in particular, we probably want the negative sign. Proceeding with this assumption, plugging in $x = \frac{3}{5}$ to the expression $y' = -\frac{1}{2} \cdot (1-x^2)^{-1/2} \cdot (-2x)$ gives $y' = -\frac{1}{2} \cdot \left[1 - \left(\frac{3}{5}\right)^2\right]^{-1/2} \cdot (-2 \cdot \frac{3}{5})$, which after some arithmetic eventually does simplify to the answer $\boxed{y' = \frac{3}{4}}$ we found above.
  - Now compare our "general" formula from the explicit solution to the problem, to the general formula to the implicit solution: the formula $y' = \pm\frac{1}{2} \cdot (1-x^2)^{1/2} \cdot (-2x)$ is far more complicated (not to mention the ambiguity about the sign) than the formula $y' = -\frac{x}{y}$.
  - In this case, we can see that the implicit formula is much better. (This is almost always the case for implicit differentiation problems!)

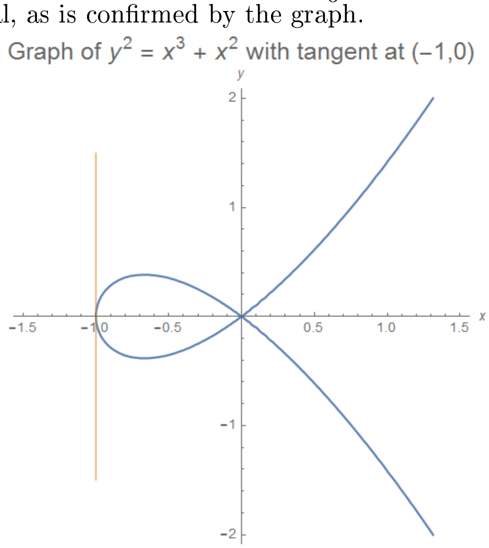
- Example: Find an equation for the tangent line to the curve $x^2 + xy + y^4 = 3$ at the point $(x, y) = (1, 1)$.
  - We have a point the tangent line passes through, namely $(1, 1)$, so all we need is the line's slope, which is the derivative y'.
  - To find y', we think of y as an implicit function $y(x)$, and write the equation as $x^2 + x \cdot [y(x)] + [y(x)]^4 = 3$.
  - Now we differentiate both sides using the Chain Rule and the Product Rule, to get $2x + (1 \cdot y(x) + x \cdot y'(x)) + 4 \cdot [y(x)]^3 \cdot y'(x) = 0$, or, more compactly, $2x + y + xy' + 4y^3y' = 0$.
  - Rearranging gives $(2x + y) + (x + 4y^3)y' = 0$, or $y' = -\frac{2x+y}{x+4y^3}$.
  - Plugging in the point $(1, 1)$ gives $y'(1, 1) = -\frac{3}{5}$.
  - Thus the tangent line has slope $-\frac{3}{5}$ and passes through $(1, 1)$, so an equation for the tangent line is $\boxed{y - 1 = -\frac{3}{5}(x-1)}$.
  - Here is a graph of the curve together with this tangent line:

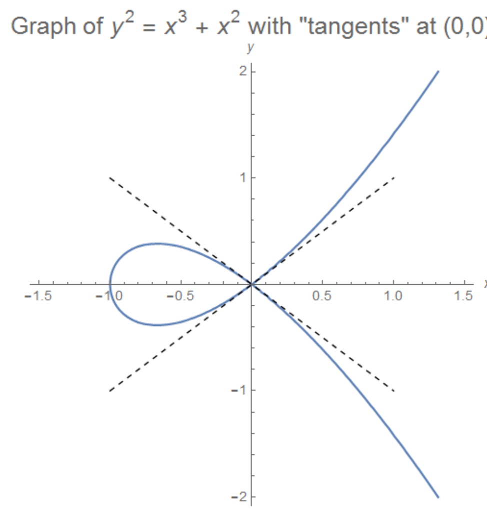


- Example: Find y' and y'' for the curve $y^2 = x^3 + x^2$. Examine what happens to y' at the point $(-1, 0)$ and at the point $(0, 0)$.
  - Differentiating both sides of $[y(x)]^2 = x^3 + x^2$ via the Chain Rule gives $2\,[y(x)] \cdot y'(x) = 3x^2 + 2x$.
  - Solving for y' gives $\boxed{y' = \dfrac{3x^2 + 2x}{2y}}$.
  - To find y'', we differentiate the expression for y' with respect to x: this yields

$$y'' = \frac{\frac{d}{dx}\left[3x^2 + 2x\right] \cdot (2y) - (3x^2 + 2x) \cdot \frac{d}{dx}\left[2y\right]}{(2y)^2} = \frac{(6x + 2) \cdot (2y) - (3x^2 + 2x) \cdot (2y')}{(2y)^2}.$$

  - Substituting in for y' gives $y'' = \dfrac{(6x + 2) \cdot (2y) - (3x^2 + 2x) \cdot \dfrac{3x^2 + 2x}{y}}{(2y)^2} = \boxed{\dfrac{(6x + 2) \cdot (2y) \cdot y - (3x^2 + 2x)^2}{4y^3}}$.
  - At $(-1, 0)$, the formula gives $y' = \dfrac{1}{0}$, which is the 'infinite' type of undefined, so here the tangent line is vertical, as is confirmed by the graph.





  - At $(0, 0)$, the formula gives $y' = \dfrac{0}{0}$, which is the 'indeterminate' type of undefined. From the graph, it appears that the curve crosses itself at the origin, and it even looks (in some sense) like there are actually two tangent lines at the origin.

- In fact, this suspicion is accurate, per the following calculation: if we solve for y, we obtain $y = x\sqrt{x+1}$ or $y = -x\sqrt{x+1}$.
- For $x \neq 0$, setting $y = x\sqrt{x+1}$ yields $y' = \dfrac{3x^2+2x}{2y} = \dfrac{3x^2+2x}{2(x\sqrt{x+1})} = \dfrac{3x+2}{2\sqrt{x+1}}$, and now taking the limit as $x \to 0$ yields $y' = 1$.
- In a similar way, setting $y = -x\sqrt{x+1}$ and then taking the limit as $x \to 0$ yields $y' = -1$.
- Thus, the two 'tangent lines' at the origin have slopes -1 and 1, meaning that they are $y = x$ and $y = -x$.
- Remark: Curves with the general form $y^2 = x^3 + a\,x^2 + b\,x + c$ are called *elliptic curves*. (The curve in this example is called a singular elliptic curve, because the graph crosses itself.) Elliptic curves have a rather surprisingly wide variety of applications: in particular, they are used extensively in cryptography (the secure transmission of information) and are the basis for some algorithms for factorization of large integers.

2.6 Parametric Curves and Derivatives

- So far, we have analyzed explicit functions of one variable (something like $y = f(x)$) and also functions defined implicitly by some kind of relation between y and x.
  - Often in physical applications, we will have systems involving a number of variables that are each a function of time, and we want to relate those variables to each other.
  - One very common situation is the following: a particle moves around in the Cartesian plane (i.e., the xy-plane) over time, so that both its x-coordinate and y-coordinate are functions of t.

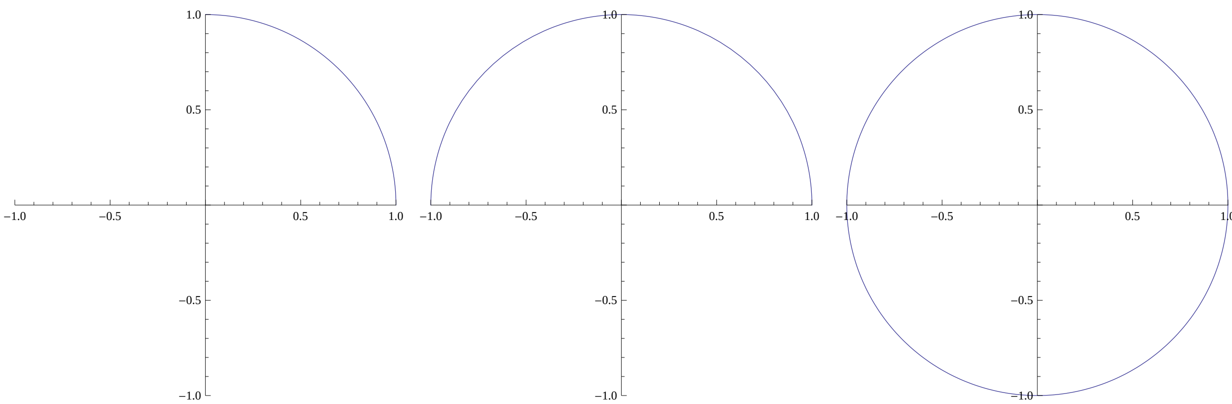
2.6.1 Parametric Curves

- Definition: If $x(t)$ and $y(t)$ are functions of t, then the set of points $(x, y) = (x(t),\, y(t))$ is called a parametric curve. The points on this curve are traced out as t varies over the real numbers.
  - Given a parametrization $x = x(t)$ and $y = y(t)$, we would like to analyze properties of the "parametric curve" that $(x(t), y(t))$ traces out in the plane, as t varies over some interval.
  - To graph a parametric curve, it is sometimes possible to find a Cartesian equation for the curve of the form $y = f(x)$. However, there is no general method for doing this.
  - In lieu of some sort of clever way to eliminate the variable t, the standard method is just to plug in many values of the parameter t to find some points $(x(t), y(t))$ on the curve, and then connect them up with a smooth curve.

- Example: Sketch the parametric curve given by $x = \cos(t)$, $y = \sin(t)$, for $0 \le t \le 2\pi$.
  - To sketch a parametric curve, we first make a table of values: we plug in easy values for t and compute the corresponding points (x, y).

t	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
x	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1
y	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

  - If we plot these points and then join them up with a smooth curve, we see that resulting shape seems to be a circle.
  - Indeed, it is a circle: from the Pythagorean identity $\sin^2(t) + \cos^2(t) = 1$, we immediately see that if $x = \cos(t)$ and $y = \sin(t)$, then $x^2 + y^2 = 1$.
  - Graphs of the parametric curve $x = \cos(t)$, $y = \sin(t)$ for $0 \le t \le \pi/2$, $0 \le t \le \pi$, and $0 \le t \le 2\pi$ are below:



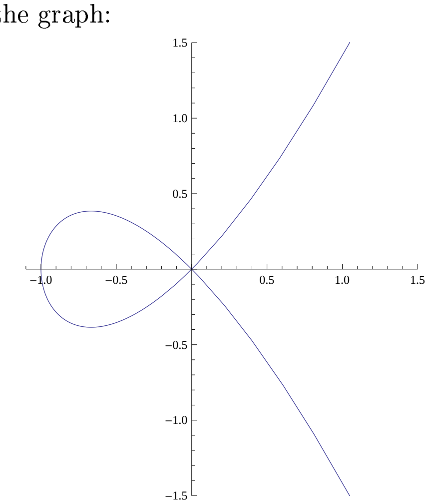
- Example: Describe the parametric curve $x = \cos(2t)$, $y = \sin(2t)$, for $0 \le t \le 2\pi$.
  - Motivated by the previous example, we see that the Pythagorean identity $\sin^2(2t) + \cos^2(2t) = 1$ again implies that $x^2 + y^2 = 1$, so this curve is once again the unit circle.
  - However, the parametrization is different: if we plug in a few values for t, we will in general not end up with the same points as before. In fact, it is easy to see that this parametrization will move along the unit circle twice as quickly as the previous one.
  - To emphasize: any curve in the plane has many different parametrizations!

- Example: Describe the parametric curve $x = a\,\cos(t)$, $y = b\,\sin(t)$, for $0 \le t \le 2\pi$, where a and b are positive real numbers.
  - From our analysis earlier, this curve will have the same general shape as a circle, but stretched by a factor of a in the x-direction and by a factor of b in the y-direction.
  - This describes an ellipse: specifically, the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. (That this is a Cartesian equation for the parametric curve is easy to see, once it is pointed out.)

- Example: Sketch the parametric curve $x = t^2 - 1$, $y = t^3 - t$ for $-\infty < t < \infty$.
  - We make a short table of values:

t	-2	-1	$-2/3$	$-1/2$	$-1/3$	0	$1/3$	$1/2$	$2/3$	1	2
x	3	0	$-5/9$	$-3/4$	$-8/9$	-1	$-8/9$	$-3/4$	$-5/9$	0	3
y	-6	0	$10/27$	$3/8$	$8/27$	0	$-8/27$	$-3/8$	$-10/27$	0	6

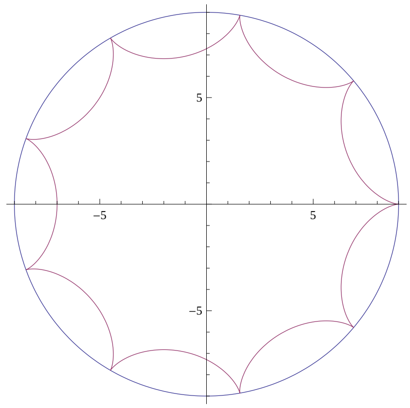
  - Using the table, we can sketch the graph:

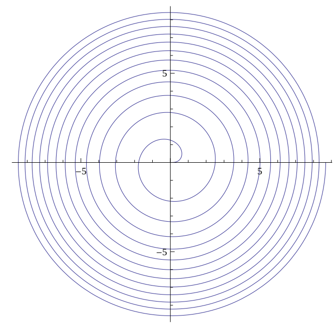


  - Notice that this curve crosses itself at the origin: it passes through the origin both when $t = -1$ and when $t = 1$.
  - This curve has a special name: it is called a (singular) elliptic curve, plotted above,

- It can be verified that this particular curve also has a Cartesian equation given by $y^2 = x^3 + x^2$.
- An elliptic curve is a curve having the general form $y^2 = x^3 + ax^2 + bx + c$; such curves have a surprisingly wide variety of applications, including in cryptography and in factoring algorithms.

- Example: Sketch the parametric curves described by (i) $x = 8\cos(t) + \cos(8t)$, $y = 8\sin(t) - \sin(8t)$, for $0 \le t \le 2\pi$, and (ii) $x = \sqrt{t}\cos(t)$, $y = \sqrt{t}\sin(t)$ for $0 \le t \le 24\pi$.
  - It is not particularly easy (or worthwhile) to plug in enough points to create accurate pictures by hand. Instead, it is best to use a computer; here are the results:





  - The first graph is called a nine-pointed hypocycloid. (A hypocycloid is a curve traced out by a point on a circle being rolled around the inside of a larger circle; to illustrate, the circle has been included in the picture.)
  - The second graph is a spiral starting at the origin and winding around counterclockwise with a continually increasing radius.

- Example: Describe the parametric curve $x = t$, $y = 0$ for $-\infty < t < \infty$, and compare it to the curve $x = \tan(t)$, $y = 0$ for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$.
  - The first curve is just the y-axis, where the particle tracing out the curve moves at constant speed.
  - The second curve is also the y-axis, but this time the particle moves at non-constant speed. (In fact, it takes only a finite amount of time to cover the entire axis.)

2.6.2 Parametric Derivatives

- Since we are doing calculus, we are naturally interested in how to compute derivatives of parametric functions.

- Parametric Differentiation Formula: The slope $\dfrac{dy}{dx}$ of the tangent line to the curve $(x(t), y(t))$ is given by $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{y'(t)}{x'(t)}$.
  - This formula is just a rearrangement of the Chain Rule $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$; solving for $\dfrac{dy}{dx}$ gives the formula.

- Example: Find the slope of the tangent line at time t for the curve parametrized by $x = \cos(t)$, $y = \sin(t)$.
  - We have $x'(t) = -\sin(t)$ and $y'(t) = \cos(t)$, so by the formula, the slope of the tangent line to the circle at $(\cos(t), \sin(t))$ is $-\cos(t)/\sin(t) = \boxed{-\cot(t)}$.
  - Remark: As we saw above, this curve is the unit circle. Note that the circle's radius has slope $y/x = \sin(t)/\cos(t) = \tan(t)$, reflecting the fact that for a circle, the radius is perpendicular to the tangent line.
  - Note also that for $t = 0$ the formula gives $-1/0$, indicating that the tangent line is vertical.

- Example: Show that the parametric curve $x = e^t + e^{-t}$, $y = e^t - e^{-t}$ traces out half of the hyperbola $x^2 - y^2 = 4$. Then find the slope of the tangent line to the curve at time t.
  - If $x = e^t+e^{-t}$ and $y = e^t-e^{-t}$, then $x^2-y^2 = (e^t+e^{-t})^2-(e^t-e^{-t})^2 = (e^{2t}+2+e^{-2t})-(e^{2t}-2+e^{-2t}) = 4$.
  - However, the parametric curve is only half of the hyperbola: $x = e^t + e^{-t}$ is always positive (since exponentials are positive), so the parametric curve only traces out the half with $x > 0$.
  - For the slope of the tangent, we have $y'(t) = e^t + e^{-t}$ and $x'(t) = e^t - e^{-t}$, so $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \boxed{\frac{e^t + e^{-t}}{e^t - e^{-t}}}$.
  - Note that since $x = e^t + e^{-t}$ and $y = e^t - e^{-t}$ we can also write $\frac{dy}{dx}$ as $\frac{x}{y}$. This is the result obtained from implicit differentiation of $x^2 - y^2 = 4$.
- In the specific situation where $(x(t), y(t))$ represents the position of a particle at time t, then $x'(t)$ is the velocity of the particle in the x-direction and $y'(t)$ is the velocity of the particle in the y-direction.
  - By putting these two quantities together[2] we obtain an expression for the velocity of the particle itself, namely, $(x'(t), y'(t))$.
  - By repeating this procedure, we also obtain an expression for the acceleration of the particle, namely, $(x''(t), y''(t))$.
- Definition: If at time t a particle has position $\mathbf{r}(t) = (x(t), y(t))$, then its velocity is defined as $\mathbf{v}(t) = (x'(t), y'(t))$ and its acceleration is defined as $\mathbf{a}(t) = (x''(t), y''(t))$. The particle's speed is the magnitude of the velocity, which is defined as $||\mathbf{v}(t)|| = \sqrt{x'(t)^2 + y'(t)^2}$.
- Example: Find the velocity, speed, and acceleration of a particle whose position at time t seconds is equal to $\mathbf{r}(t) = (2 + e^{2t}, 2 - e^{2t})$ meters.
  - Since $x(t) = 2 + e^{2t}$ meters and $y(t) = 2 - e^{2t}$ meters , we have $x'(t) = 2e^{2t}\text{m/s}$ and $y'(t) = -2e^{2t}\text{m/s}$, so the velocity is $\mathbf{v}(t) = \boxed{(2e^{2t}, -2e^{2t})\text{m/s}}$.
  - Then the speed is $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(2e^{2t}\text{m/s})^2 + (-2e^{2t}\text{m/s})^2} = \boxed{2\sqrt{2}e^{2t}\text{m/s}}$.
  - Likewise, since $x''(t) = 4e^{2t}$ and $y''(t) = -4e^{2t}$, the acceleration is $\mathbf{a}(t) = \boxed{(4e^{2t}, -4e^{2t})}$.

2.7 Linearization and Differentials

- Another use of derivatives is approximating complicated functions by simpler ones. Such approximations are particularly common in physics and economics, where they can allow one to analyze complicated systems that are otherwise very difficult to study.
  - In particular, linear approximations are useful when the relation between two variables can only be written implicitly, and we would like to be able to say something about what happens to one variable if the other changes.

2.7.1 Linearization, Approximation by the Tangent Line

- The key insight is: the tangent line to the graph of a function is a good approximation to the function near the point of tangency. Here is a justification of this idea:
  - The definition of the derivative says that $f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$. (Note that Δ is the Greek letter delta, and is intended to indicate a small change in the variable following it.)

[2] More properly, we are actually treating $(x(t), y(t))$ as a vector, rather than an ordered pair representing the coordinates of a point in the plane.

- Intuitively, the definition says that for small values of Δx, the quotient $\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ should be close to the value $f'(x_0)$, which we write as $\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \approx f'(x_0)$. (The symbol $\approx$ means "approximately equal to".)
- Clearing the denominator and rearranging yields $f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\cdot\Delta x$, where Δx is small.
- Equivalently, if we write $x = x_0 + \Delta x$, this says $f(x) \approx f(x_0) + f'(x_0)\cdot(x - x_0)$, where $x - x_0$ is small (i.e., when x is close to x_0).
- Note how similar this expression is to the equation of the tangent line to $y = f(x)$ at $x = x_0$, which is $y = f(x_0) + f'(x_0)\cdot(x - x_0)$.

- Definition: If $f(x)$ is differentiable at $x = x_0$, then the linearization of $f(x)$ at $x = x_0$ is defined to be the linear function $L(x) = f(x_0) + f'(x_0)\cdot(x - x_0)$.
- Example: Find the linearization of the function $(1+x)^3$ at $x = 0$.
  - We have $f(x) = (1+x)^3$ and $x_0 = 0$, and also $f'(x) = 3(1+x)^2$ by the Chain Rule.
  - Applying the formula shows $L(x) = f(0) + f'(0)\cdot(x-0) = \boxed{1+3x}$.
- The reason the linearization is useful is because it is the best linear approximation to a differentiable function:
- Theorem (Linear Approximation): If $f(x)$ is differentiable at $x = x_0$, then the linearization of $f(x)$ near $x = x_0$, defined by $L(x) = f(x_0) + f'(x_0)\cdot(x - x_0)$, is the only linear function satisfying $\lim\limits_{x\to x_0} \dfrac{f(x) - L(x)}{x - x_0} = 0$.
  - Equivalently, this result says that the tangent line $y = L(x)$ is the only function with the property that, when we approximate $f(x)$ with $L(x)$, the resulting error near $x = x_0$ is small when compared to $x - x_0$.
  - Proof: Suppose $f(x)$ is differentiable at $x = x_0$.
  - We compute $\lim\limits_{x\to x_0} \dfrac{f(x)-L(x)}{x-x_0} = \lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)-f'(x_0)\cdot(x-x_0)}{x-x_0} = \lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0} - f'(x_0) = 0$ by the definition of the derivative, so the linearization does have the desired property.
  - Now suppose that $g(x) = a + b(x - x_0)$ is another linear function with $\lim\limits_{x\to x_0} \dfrac{f(x)-g(x)}{x-x_0} = 0$.
  - By the limit laws, $0\cdot 0 = \lim\limits_{x\to x_0} \dfrac{f(x)-g(x)}{x-x_0} \cdot \lim\limits_{x\to x_0}(x-x_0) = \lim\limits_{x\to x_0}(f(x)-g(x)) = f(x_0) - g(x_0)$, so $a = f(x_0)$.
  - Also, $0 = \lim\limits_{x\to x_0} \dfrac{f(x)-g(x)}{x-x_0} = \lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)-b(x-x_0)}{x-x_0} = \lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0} - b = f'(x_0) - b$, so $b = f'(x_0)$.
  - So we see that the linearization is the only possible $g(x)$, as claimed.
- We can use the linearization of a function to find numerical approximations to function values.
  - The idea is to find a function f, a nice value of x_0, and a small Δx such that $f(x_0)$ is easy to compute and $f(x_0 + \Delta x)$ is the value we want.
  - Then $f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\cdot\Delta x$ gives the approximate value desired.
- Example: Find a linear approximation to $f(x) = \ln(x)$ near $x = 1$, and use it to approximate $\ln(0.9)$.
  - For $f(x) = \ln(x)$ we have $f'(x) = 1/x$.
  - Thus, the linearization near $x = 1$ is $L(x) = \ln(1) + (x-1)\cdot 1 = \boxed{x-1}$.
  - Then $\ln(0.9) \approx L(0.9) = \boxed{-0.1}$.
  - With a calculator, we compute $\ln(0.9) \approx -0.10536\ldots$, so the linear approximation is quite good.
- Example: Approximate the values of $\sqrt{1.1}$ and $\sqrt{1.4}$.

- To approximate $\sqrt{1.1}$ the most natural thing to try $x_0 = 1$ and $\Delta x = 0.1$, and $f(x) = \sqrt{x}$: then $f(x_0 + \Delta x) = \sqrt{1.1}$ is the value we want to compute.
- For $f(x) = \sqrt{x} = x^{1/2}$ we have $f'(x) = \frac{1}{2} x^{-1/2}$.
- We get the approximation $\sqrt{1.1} = f(1 + 0.1) \approx \sqrt{1} + \frac{1}{2} \cdot 1^{-1/2} \cdot (0.1) = 1 + \frac{1}{2} \cdot 0.1 = \boxed{1.05}$.
- With a computer, we obtain $\sqrt{1.1} = 1.04881\ldots$, so we see that the linear approximation is quite good.
- If instead we take $\Delta x = 0.4$, we get the approximation $\sqrt{1.4} = f(1 + 0.4) \approx 1 + \frac{1}{2} \cdot 1^{-1/2} \cdot (0.4) = \boxed{1.2}$.
- With a computer, we obtain $\sqrt{1.4} = 1.18322\ldots$. The approximation is still good, but not as good as the previous one.

- Although the linearization gives the best possible linear approximation to a differentiable function, we have not actually given any way to estimate how large the approximation error might be.
  - If we do not actually know how accurate the approximation is, then it will not be particularly helpful for doing calculations.
  - It is possible to give an error estimate for the linearization (presuming the function is twice-differentiable), but we will postpone discussion of this topic for now since we must develop more background first.
  - For completeness, here is such an error estimate: if $f(x)$ is a function whose second derivative is continuous, then for any values a and b, if $L_a(x)$ is the linearization of f at $x = a$, then $|f(b) - L_a(b)| \le M \cdot \frac{|b-a|^2}{2}$ where M is any constant such that $\left|f''(x)\right| \le M$ for all x in the interval $[a, b]$.

- Remark: There is no reason only to consider approximation by linear functions, aside from the fact that linear functions are the easiest to analyze. We could just as well look for quadratic approximations $a_0 + a_1 x + a_2 x^2$, or cubic approximations $a_0 + a_1 x + a_2 x^2 + a_3 x^3$, or general polynomial approximations to $f(x)$.
  - This is precisely the idea behind "Taylor polynomials", which (among other things) give even better approximations of a function than the linearization does. However, developing these ideas fully requires integral calculus, so we will not pursue the discussion further at the moment.
  - It is also possible to use different classes of functions to approximate functions, such as trigonometric functions. Approximating a function by one of the form $a_0 + a_1 \sin(x) + b_1 \cos(x) + a_2 \sin(2x) + b_2 \cos(2x) + \cdots$ leads to the idea of a Fourier series, which possess an equally deep and useful theory (and which, unfortunately, we also will not develop further at the moment).

2.7.2 Differentials

- Definition: When $y = f(x)$ is given as a function of x, the differential dy is defined as $dy = f'(x)\,dx$, where dx is an independent variable.
  - As can be seen by rearranging the definition, we have defined differentials so that the derivative, which we have denoted as $f'(x) = \frac{dy}{dx}$, is now actually a quotient of variables.
  - The intuitive idea of differentials is that as Δx tends to zero the approximate equality $\Delta y \approx f'(x_0) \cdot \Delta x$ becomes more and more accurate. "In the limit" we turn the "Δ" symbols into "d" symbols.

- Differentials are a piece of what is sometimes (pejoratively) called "abstract nonsense": some theoretical notation which is useful and yields correct answers, but whose rigorous use we do not have the tools to justify at this stage.
  - There are several different technical ways to make sense of differentials, but all of them involve significantly more advanced mathematics than calculus.
  - Our primary interest in differentials is that they greatly simplify some computations and arguments in integral calculus, particularly in calculations involved in changing variables.

- Differentials also simplify and make clearer some of the properties of derivatives: for example, the Chain Rule's differential form $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$ can now be interpreted as a "cancellation" of the dy terms.

- Example: If $y = x^2$, then we have the differential $\boxed{dy = f'(x)\,dx = 2x\,dx}$.

- Example: If $y = x^3$ and $z = y^2$, find dz in terms of dx.
  - By the definition we have $dz = 2y\,dy$ and $dy = 3x^2\,dx$.
  - Plugging the second into the first gives $dz = 2(x^3)(3x^2 dx) = \boxed{6x^5\,dx}$.
  - Alternatively, we could have written $z = y^2 = (x^3)^2 = x^6$, and then directly computed the differential $dz = \boxed{6x^5\,dx}$.
  - Note that we obtained the same result either way, on account of the Chain Rule.

2.8 Related Rates

- If we have a relation between two quantities, then by differentiating that relation and applying the Chain Rule we obtain a relationship between the derivatives of these quantities.
  - Thus, if we know the rate that one quantity is changing, we can find the rate at which the other is changing. Problems of this type are called "related rates" problems, as they involve analyzing the rate of change of one quantity using information about related quantities.

- To solve a related rates problem, follow these steps:
  - First, if not already given, choose variable names and translate the given information into mathematical statements. Identify all relations between the variables. Draw a picture, if the relations are not immediately clear.
  - Next, differentiate each relation using the Chain Rule.
  - Finally, solve for the desired quantity using the given information and the information obtained from the derivatives.

- Example: A spherical balloon is inflated such that its volume increases by $40\pi\,\text{cm}^3$ per second. Find the rate at which the balloon's radius is increasing (i) when the radius is 2 cm, and (ii) when the radius is 5 cm.
  - The volume of the balloon is given by $V = \frac{4}{3}\pi r^3$ (in cm^3), where r is the radius of the balloon (in cm).
  - We are given that $\frac{dV}{dt} = 40\pi\,\text{cm}^3/\text{s}$ and want to find $\frac{dr}{dt}$.
  - Differentiating the relation $V = \frac{4}{3}\pi r^3$ via the Chain Rule gives $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.
  - Then, solving for $\frac{dr}{dt}$ gives $\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$. Now we can plug in the appropriate values to find the desired information.
  - When $r = 2$ we see $\frac{dr}{dt} = \frac{40\pi\,\text{cm}^3/\text{s}}{4\pi \cdot 2^2\,\text{cm}^2} = \frac{5}{2}\text{cm}/\text{s}$. So when $r = 2$, the radius is increasing at $\boxed{2.5\,\text{cm}/\text{s}}$.
  - When $r = 5$ we see $\frac{dr}{dt} = \frac{40\pi\,\text{cm}^3/\text{s}}{4\pi \cdot 5^2\,\text{cm}^2} = \frac{2}{5}\text{cm}/\text{s}$. So when $r = 5$, the radius is increasing at $\boxed{0.4\,\text{cm}/\text{s}}$.
  - Note that we obtain the correct units (cm/s, distance per time) for the rate of change $\frac{dr}{dt}$.

- Example: Two ships leave a port, one traveling north and the other traveling east. After 4 hours, the north-moving ship is moving at 40 km/h and is 120 km from the port, and the east-moving ship is moving at 30 km/h and is 90 km from the port. Find the rate at which the distance between the ships is changing after 4 hours.

- Let the distance of the east-traveling ship from the port be $x(t)$ and the distance of the north-traveling ship from the port be $y(t)$, where t is measured in hours.
- Then by the Pythagorean Theorem, the distance $d(t)$ between the ships satisfies $d(t)^2 = x(t)^2 + y(t)^2$. We want to find $d'(t)$.
- The given information says that $x(4) = 120\,\text{km}$, $x'(4) = 40\,\text{km/h}$, $y(4) = 90\,\text{km}$, and $y'(4) = 30\,\text{km/h}$.
- Plugging in $t = 4$ gives $d(4)^2 = x(4)^2 + y(4)^2 = (120\,\text{km})^2 + (90\,\text{km})^2 = 150^2\text{km}^2$, so $d(4) = 150\,\text{km}$.
- Now, differentiating the relation $d(t)^2 = x(t)^2 + y(t)^2$ via the Chain Rule gives $2 \cdot d(t) \cdot d'(t) = 2 \cdot x(t) \cdot x'(t) + 2 \cdot y(t) \cdot y'(t)$.
- Upon solving for the desired quantity $d'(t)$, we see $d'(t) = \dfrac{x(t) \cdot x'(t) + y(t) \cdot y'(t)}{d(t)}$.
- If we then take $t = 4$ and plug in all of the known values, we get $d'(4) = \dfrac{x(t) \cdot x'(t) + y(t) \cdot y'(t)}{d(t)} = \dfrac{(120\,\text{km})(40\text{km/h}) + (90\,\text{km})(30\text{km/h})}{150\,\text{km}} = \dfrac{7500\text{km}^2\text{/h}}{150\,\text{km}} = \boxed{50\,\text{km/h}}$.

- Example: A large rectangular box has a length of 6 m, a width of 4 m, and a height of 3 m. The length is changing at a rate of +1 m/s, the width is changing at a rate of −2 m/s, and the height is changing at a rate of +1 m/s. Find the rates at which the volume and total surface area of the crate are changing.
  - Let the length be $l(t)$, the width be $w(t)$, and the height be $h(t)$.
  - We are given $l = 6\,\text{m}$, $w = 4\,\text{m}$, $h = 3\,\text{m}$, $l' = +1\text{m/s}$, $w' = -2\text{m/s}$, and $h' = +1\text{m/s}$, all at time $t = 0$.
  - The volume is given by $V = l \cdot w \cdot h$ and the surface area is given by $A = 2lw + 2lh + 2wh$.
  - Differentiating each expression yields

$$\frac{dV}{dt} = \frac{d}{dt}\left[(lw) \cdot h\right] = \frac{d}{dt}\left[lw\right] \cdot h + (lw) \cdot \frac{dh}{dt} = \left(\frac{dl}{dt} \cdot w + l \cdot \frac{dw}{dt}\right) \cdot h + (lw) \cdot \frac{dh}{dt} = \frac{dl}{dt}\,wh + l\,\frac{dw}{dt}\,h + lw\,\frac{dh}{dt}$$

and

$$\frac{dA}{dt} = \frac{d}{dt}\left[2lw + 2lh + 2wh\right] = 2\left[\frac{dl}{dt}\,w + l\,\frac{dw}{dt} + \frac{dl}{dt}\,h + l\,\frac{dh}{dt} + \frac{dw}{dt}\,h + w\,\frac{dh}{dt}\right].$$

  - Now we just need to plug in all of the given data. We obtain

$$V' = (1\text{m/s})(4\,\text{m})(3\,\text{m}) + (6\,\text{m})(-2\text{m/s})(3\,\text{m}) + (6\,\text{m})(4\,\text{m})(1\text{m/s}) = 0\text{m}^3\text{/s}$$

and

$$\begin{aligned} A' &= 2 \cdot \left[(1\text{m/s})(4\,\text{m}) + (6\,\text{m})(-2\text{m/s}) + (+1\text{m/s})(3\,\text{m}) + (6\,\text{m})(1\text{m/s}) + (-2\text{m/s})(3\,\text{m}) + (4\,\text{m})(+1\text{m/s})\right] \\ &= 2 \cdot \left[4\text{m}^2\text{/s} - 12\text{m}^2\text{/s} + 3\text{m}^2\text{/s} + 6\text{m}^2\text{/s} - 6\text{m}^2\text{/s} + 4\text{m}^2\text{/s}\right] = 2\text{m}^2\text{/s}. \end{aligned}$$

  - So the volume of the box is $\boxed{\text{not changing}}$, and the surface area is $\boxed{\text{increasing at } 2\,\text{m}^2\text{/s}}$.

Well, you're at the end of my handout. Hope it was helpful.