Calculus I (part 1s): Formal ϵ-δ Limits (by Evan Dummit, 2012, v. 1.55)

The formal use of ϵ-δ limits does not yield a great deal of insight into how to use limits at a beginning level. As such, the development of the formal theory of limits has been relegated to this supplement. Or, more dramatically:

DO NOT READ THIS SUPPLEMENT UNLESS YOU ABSOLUTELY HAVE TO

Contents

0.0.1 Formal Definition and Examples

- Definition: A function $f(x)$ has the limit L as $x \to a$, written as $\lim_{x \to a} f(x) = L$, if, for any $\epsilon > 0$ (no matter how small) there exists a $\delta > 0$ (depending on ϵ) with the property that for all $0 < |x - a| < \delta$, we have that $|f(x) - L| < \epsilon$.
  - The symbols δ and ϵ are the lowercase Greek letters delta and epsilon (respectively). Their use in the definition of the limit is traditional. Also recall that the notation $|x|$ means the absolute value of x, and denotes the distance from x to zero.
  - One way to think of this definition is as follows: suppose you claim that the function $f(x)$ has a limit L, as x gets close to a. In order to prove to me that the function really does have that limit, I challenge you by handing you some value $\epsilon > 0$, and I want you to give me some open interval $(a - \delta, a + \delta)$ on the x-axis containing a, with the property that $f(x)$ is always within ϵ for x in that interval, except possibly at a.
  - If $f(x)$ really does stay close to the limit value L as x gets close to a, then, no matter what value of ϵ I picked, you should always be able to answer my challenge with an interval around a, because the values of $f(x)$ should stay near L when x is near a.
  - Note that it is not necessary to find the best possible δ – any δ which does the job is perfectly fine.
- Important Remark: Don't worry if this formal definition seems very opaque at first. It takes practice and experience to become comfortable with what the definition means, and to see why it really does match the intuition of how a limit should behave.
  - We will generally use the formal definition primarily as a tool to justify our manipulations of limits and to ground our intuition in proof.
- Basic Limit 1: $\lim_{x \to a} c = c$, where c is any constant.
  - To show this formally, suppose we are given an $\epsilon > 0$, and we want to find a $\delta > 0$ which will make $|c - c| < \epsilon$ whenever $0 < |x - a| < \delta$.
  - Now, since $|c - c| = 0$, and $\epsilon > 0$, the inequality $|c - c| < \epsilon$ is always true.
  - So in fact here we can take any δ we want – any at all – and the result holds.
- Basic Limit 2: $\lim_{x \to a} x = a$.

- ◦ Proof: Suppose we are given an $\epsilon > 0$, and we want to find a $\delta > 0$ which will make $|h(x) - a| < \epsilon$ whenever $0 < |x - a| < \delta$, where $h(x) = x$.
  - ◦ Let's try taking our δ to equal ϵ. Then we need to check that $0 < |x - a| < \delta$ makes $|x - a| < \epsilon$.
  - ◦ This is in fact true, since if $0 < |x - a| < \epsilon$ then certainly $|x - a| < \epsilon$. So this choice of δ works, and so the limit has the value we claimed.
  - ◦ Note that we could have chosen δ to be lots of other things, and it still would have worked: for example, $\delta = \epsilon/2$ would also have worked.

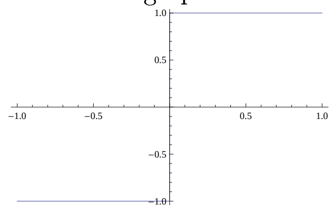
- In that example it may have seemed like we just guessed that we should try $\delta = \epsilon$, and then plugged in to see that it would work. In general, this is how formal limit proofs work – generally, one needs some insight or observation to figure out what δ to use, but all that is needed for the proof is to plug in to see that the choice of δ actually works.
  - ◦ For arbitrary functions, there is not any general rule for finding δ from ϵ.
  - ◦ Often what is needed is to try to solve the problem "backwards": i.e., to rearrange the wanted inequality $|f(x) - L| < \epsilon$ using properties of the function $f(x)$, to figure out what value of δ will make things work.

- Example: We prove that $\lim_{x \to 3} x^2 = 9$.
  - ◦ We are given $\epsilon > 0$ and want to pick δ such that $|x - 3| < \delta$ implies $\left|x^2 - 9\right| < \epsilon$.
  - ◦ We can factor $(x^2 - 9) = (x - 3)(x + 3)$, and so we are looking to pick δ which makes $|x - 3| \cdot |x + 3| = |(x - 3)(x + 3)| < \epsilon$.
  - ◦ Our hypothesis $|x - 3| < \delta$ already tells us that we can make $|x - 3|$ small, but what about the other term in that product, $|x + 3|$?
  - ◦ We are free to choose δ however we like, so (for instance) we can always insist on taking $\delta \le 1$. Then since $|x - 3| < \delta \le 1$, this says $2 < x < 4$, and so $5 < x + 3 < 7$. So we can say that $|x + 3|$ is always less than 7.
  - ◦ Thus, if we choose $\delta \le 1$, we know that $|x - 3| \cdot |x + 3| < |x - 3| \cdot 7 < 7\delta$.
  - ◦ If we can make this always less than ϵ, we will be done. So, for instance, we can take $\delta = \epsilon/7$ to make this work. Remembering the condition $\delta \le 1$, we conclude that taking $\delta = \min(1,\ \epsilon/7)$ should always work.
  - ◦ If we wanted to write this up carefully, we would do it as follows:
    - ∗ We claim that $\lim_{x \to 3} x^2 = 9$.
    - ∗ We are given $\epsilon > 0$ and want to find $\delta > 0$ such that $|x - 3| < \delta$ implies $\left|x^2 - 9\right| < \epsilon$.
    - ∗ We claim that $\delta = \min(1,\ \epsilon/7)$ will always work.
    - ∗ To show this works, suppose that$|x - 3| < \delta$.
      - · In particular, since $\delta \le 1$ we see that$|x - 3| < 1$, so $-1 < x - 3 < 1$.
      - · Adding 6 to each part of this inequality gives $5 < x + 3 < 7$, and so $|x + 3| < 7$.
      - · Finally, $\left|x^2 - 9\right| = |x - 3| \cdot |x + 3| < \delta \cdot 7 < 7 \cdot (\epsilon/7) = \epsilon$, as we wanted.
    - ∗ Therefore, $\lim_{x \to 3} x^2 = 9$.
  - ◦ When we write the proof up this way, it looks very clean, but it obscures all the work we had to do in order to figure out what δ actually should be. This will usually be the case with formal $\epsilon - \delta$ proofs: they often seem like merely an excellent guess.

- We can also use this definition to prove that a limit does not exist. Showing that a given limit does not exist (at all) is the same as showing that no matter what value of L we pick, there exists some "bad" value of ϵ for which we can't find any δ.

- Example: Show that the step function $\frac{x}{|x|}$, which has value -1 for negative x, $+1$ for positive x, and is undefined at 0, has no limit as $x \to 0$.

◦ Here is the graph of this function:



◦ To show the function has no limit at $x = 0$, we need to verify that that no matter what value of L we pick, there is a value of ϵ which falsifies the limit hypothesis.

◦ Here, we can pick $\epsilon = 1/3$ no matter what value of L we're trying:

* Since any open interval around 0 contains both positive and negative numbers, we would need both $|1 - L|$ and $|-1 - L|$ to be less than $\epsilon = 1/3$.
* But the sum $|1 - L| + |-1 - L|$ is always at least 2: if L is between -1 and $+1$ then the sum is 2 and otherwise the sum is $2|L|$.
* This is obviously impossible, since the sum of two things each less than 1/3 cannot be at least 2.
* Therefore, the limit cannot be L, for any value of L. In other words, the limit does not exist.

◦ Remark: One can make a similar argument for $f(x) = \cos\left(\frac{1}{x}\right)$ to see that it has no limit at $x = 0$.

* In any interval around 0, no matter how small, there are points where $f(x) = 1$ and $f(x) = -1$.
* Essentially the same argument as above (with $\epsilon = 1/3$) will show that $\cos\left(\frac{1}{x}\right)$ has no limit at $x = 0$.

0.0.2 Proofs of the Limit Rules for Finite Limits

• Let $f(x)$ and $g(x)$ be functions satisfying $\lim_{x \to a} f(x) = L_f$ and $\lim_{x \to a} g(x) = L_g$. Then the following properties hold:

• The addition rule: $\lim_{x \to a} [f(x) + g(x)] = L_f + L_g$.

◦ Proof: Suppose we are given $\epsilon > 0$.

* Since we know that $\lim_{x \to a} f(x) = L_f$ and $\lim_{x \to a} g(x) = L_g$, we can find δ_1 and δ_2 such that $|f(x) - L_f| < \frac{\epsilon}{2}$ for $0 < |x - a| < \delta_1$ and $|g(x) - L_g| < \frac{\epsilon}{2}$ for $0 < |x - a| < \delta_2$.
* We claim that the value $\delta = \min(\delta_1, \delta_2)$ will make $|f(x) + g(x) - (L_f + L_g)| < \epsilon$ for all x with $0 < |x - a| < \delta$.
* To verify: we know that $-\frac{\epsilon}{2} < f(x) - L_f < \frac{\epsilon}{2}$ and $-\frac{\epsilon}{2} < g(x) - L_g < \frac{\epsilon}{2}$ for $0 < |x - a| < \min(\delta_1, \delta_2)$, so adding the inequalities shows $-\epsilon < (f(x) - L_f) + (g(x) - L_g) < \epsilon$.
* Or, in other words, $|f(x) + g(x) - (L_f + L_g)| < \epsilon$ for all x with $0 < |x - a| < \delta$. This is what we wanted to show.

• The subtraction rule: $\lim_{x \to a} [f(x) - g(x)] = L_f - L_g$.

◦ Proof: The same as the addition rule, but with a minus sign instead of a plus sign.

• The multiplication rule: $\lim_{x \to a} [f(x) \cdot g(x)] = L_f \cdot L_g$.

◦ Proof: First we show that if $\lim_{x \to a} h(x) = L_h$ then $\lim_{x \to a} h(x)^2 = L_h^2$.

* Suppose we are given $\epsilon > 0$.
* Since we know that $\lim_{x \to a} h(x) = L_h$, then we can find δ such that $|h(x) - L_h| < \min\left(1, \frac{\epsilon}{1 + 2|L_h|}\right)$ for all $0 < |x - a| < \delta$.

* In particular, for $0 < |x-a| < \delta$ we have $|h(x) - L_h| < 1$ or $-1 < h(x) - L_h < 1$, so that $-1 + 2L_h < h(x) + L_h < 1 + 2L_h$. Taking absolute values shows $|h(x) + L_h| < 1 + 2|L_h|$.
* Then $\left|h(x)^2 - L_h^2\right| = |h(x) - L_h| \cdot |h(x) + L_h| < \dfrac{\epsilon}{1 + 2|L_h|} \cdot (1 + 2|L_h|) = \epsilon$, as desired.

  - Now, applying this result to the two particular cases where $h(x) = f(x) + g(x)$ and $h(x) = f(x) - g(x)$ shows (after an application of the addition rule and the subtraction rule) that $\lim\limits_{x\to a}[f(x) + g(x)]^2 = (L_f + L_g)^2$ and $\lim\limits_{x\to a}[f(x) - g(x)]^2 = (L_f - L_g)^2$.
  - Subtracting, applying the subtraction rule again, and cancelling the common terms gives $\lim\limits_{x\to a}[4f(x) \cdot g(x)] = 4L_f \cdot L_g$. Dividing by 4 then gives the multiplication rule.

- The division rule: $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{L_f}{L_g}$, provided that L_g is not zero.
  - Proof: First we show that if $\lim\limits_{x\to a} h(x) = L_h$ and $L_h > 0$ then $\lim\limits_{x\to a} \dfrac{1}{h(x)} = \dfrac{1}{L_h}$. Suppose we are given $\epsilon > 0$.
    * Since we know that $\lim\limits_{x\to a} h(x) = L_h$, then we can find δ such that $|h(x) - L_h| < \min\left(\dfrac{L_h}{2}, 2L_h^2\epsilon\right)$ for all $0 < |x-a| < \delta$.
    * In particular, for $0 < |x-a| < \delta$ we have $|h(x) - L_h| < \dfrac{L_h}{2}$, so that $\dfrac{L_h}{2} < h(x) < \dfrac{3L_h}{2}$ and hence $|h(x)| < \dfrac{3L_h}{2}$.
    * Then $\left|\dfrac{1}{h(x)} - \dfrac{1}{L_h}\right| = \left|\dfrac{L_h - h(x)}{h(x) \cdot L_h}\right| = |h(x) - L_h| \cdot \dfrac{1}{|L_h|} \cdot \dfrac{1}{|h(x)|} < (2L_h^2\epsilon) \cdot \dfrac{1}{L_h} \cdot \dfrac{1}{\frac{1}{2}L_h} = \epsilon$, as desired.
  - If $L_h < 0$ then the same result holds by the subtraction rule.
  - Finally, to obtain the division rule, we apply the multiplication rule to $f \cdot \dfrac{1}{g}$.

- The exponentiation rule: $\lim\limits_{x\to a}[f(x)^a] = (L_f)^a$, where a is any positive real number. (It also holds when a is negative or zero, provided L_f is positive, in order for both sides to be real numbers.)
  - Proof: If a is a positive integer, we repeatedly apply the multiplication rule to $f^n = f^{n-1} \cdot f$.
  - For positive rational $a = \dfrac{p}{q}$ we write $[f^a]^q = f^p$ and then apply the integer case of the exponentiation rule.
  - If a is negative, we apply the division rule to $\dfrac{1}{f(x)^a}$ and apply the case where a is positive.
  - For general real numbers a, the proof requires the definition of $f(x)^a$ as a limit of a sequence.

- The inequality rule: If $f(x) \le g(x)$ for all x, then $L_f \le L_g$.
  - Proof: Suppose by way of contradiction that $\lim\limits_{x\to a} f(x) = L_f$ and $\lim\limits_{x\to a} g(x) = L_g$, where $L_g < L_f$.
    * Denote the positive number $L_f - L_g$ by α.
    * Because $\lim\limits_{x\to a} f(x) = L_f$, by definition there exists a δ_1 for which $|f(x) - L_f| < \dfrac{\alpha}{3}$ for all $0 < |x-a| < \delta_1$.
    * Similarly, because $\lim\limits_{x\to a} f(x) = L_f$, there exists a δ_2 for which $|g(x) - L_g| < \dfrac{\alpha}{3}$ for all $0 < |x-a| < \delta_2$.
    * Now pick any value y with $0 < |y-a| < \min(\delta_1, \delta_2)$: for that y, we have $|f(y) - L_f| < \dfrac{\alpha}{3}$ and $|g(y) - L_g| < \dfrac{\alpha}{3}$.
    * Then in particular, we have $g(y) < L_g + \dfrac{\alpha}{3} < L_g + \dfrac{2\alpha}{3} = L_f - \dfrac{\alpha}{3} < f(y)$.

    * This is a contradiction because we have $g(y) < f(y)$, but we assumed that $f(x) \le g(x)$ for all x.

- The squeeze rule (also called the sandwich rule): If $f(x) \le g(x) \le h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ (meaning that both limits exist and are equal to L) then $\lim_{x \to a} g(x) = L$ as well.
  - Proof: Suppose we are given $\epsilon > 0$.
    * Since we know that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} h(x) = L$, we can find δ_1 and δ_2 such that $|f(x) - L| < \epsilon$ for $0 < |x - a| < \delta_1$ and $|g(x) - L| < \epsilon$ for $0 < |x - a| < \delta_2$.
    * We claim that the value $\delta = \min(\delta_1, \delta_2)$ will make $|g(x) - L| < \epsilon$ for all x with $0 < |x - a| < \delta$.
    * To verify: we know that $-\epsilon < f(x) - L$ and $h(x) - L < \epsilon$ for $0 < |x - a| < \min(\delta_1, \delta_2)$, so by adding L to both inequalities we see $L - \epsilon < f(x)$ and $h(x) < L + \epsilon$.
    * Since $f(x) \le g(x) \le h(x)$ for all x, we see that $L - \epsilon < f(x) \le g(x) \le h(x) < L + \epsilon$ for $0 < |x - a| < \delta$. In other words, $L - \epsilon < g(x) < L + \epsilon$, or $|g(x) - L| < \epsilon$.
    * So $|g(x) - L| < \epsilon$ for all x with $0 < |x - a| < \delta$. This is what we wanted to show.

- Proposition: A two-sided limit exists if and only if both one-sided limits exist and have the same value.
  - Proof: Let $\epsilon > 0$ be given. If the two-sided limit exists, it is clear that both one-sided limits exist and have the same value, since we can use the value of δ from the two-sided limit in each of the one-sided limits.
  - Conversely, suppose that $\lim_{x \to a+} f(x) = L = \lim_{x \to a-} f(x)$. Then from the left limit there exists a δ_1 such that $-\delta_1 < x - a < 0$ implies $|f(x) - L| < \epsilon$, and from the right limit there exists a δ_2 such that $0 < x - a < \delta_2$ implies $|f(x) - L| < \epsilon$.
  - Hence for $\delta = \min(\delta_1, \delta_2)$, we see that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$, as desired.

- Theorem: If $g(x)$ is a continuous function, and $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} g(f(x)) = g(L)$.
  - Proof: Let $\epsilon > 0$ be given. By hypothesis, since g is continuous there exists $\delta_1 > 0$ for which $|g(y) - g(L)| < \epsilon$ for all y with $|y - L| < \delta_1$.
  - Now since $\lim_{x \to a} f(x) = L$, there exists $\delta_2 > 0$ for which $|f(x) - L| < \delta_1$ for all x with $0 < |x - a| < \delta_2$.
  - Plugging the second statement into the first shows that $|g(f(x)) - g(L)| < \epsilon$ for all x with $0 < |x - a| < \delta_2$, as desired.

0.0.3 Formal Arguments for Infinite Limits

- The formal definitions for one-sided limits are the following:
  - We say that a function $f(x)$ has the limit L as $x \to a$ from below, written as $\lim_{x \to a-} f(x) = L$, if the following statement is true: for any $\epsilon > 0$ (no matter how small) there exists a $\delta > 0$ (depending on ϵ) with the property that for all $0 < a - x < \delta$, we have that $|f(x) - L| < \epsilon$.
  - We say that a function $f(x)$ has the limit L as $x \to a$ from above, written as $\lim_{x \to a+} f(x) = L$, if the following statement is true: for any $\epsilon > 0$ (no matter how small) there exists a $\delta > 0$ (depending on ϵ) with the property that for all $0 < x - a < \delta$, we have that $|f(x) - L| < \epsilon$.

- The formal definitions for limits at infinity are the following:
  - We say that a function $f(x)$ has the limit L as $x \to +\infty$, written as $\lim_{x \to +\infty} f(x) = L$, if the following statement is true: for any $\epsilon > 0$ (no matter how small) there exists an $A > 0$ (depending on ϵ) with the property that for all $x > A$, we have that $|f(x) - L| < \epsilon$.
  - We say that a function $f(x)$ has the limit L as $x \to -\infty$, written as $\lim_{x \to -\infty} f(x) = L$, if the following statement is true: for any $\epsilon > 0$ (no matter how small) there exists an $A > 0$ (depending on ϵ) with the property that for all $x < -A$, we have that $|f(x) - L| < \epsilon$.

- Note: As ϵ shrinks, A will grow very large, in contrast to when we used δ, which would get smaller with smaller ϵ.
- Notation: The symbols ∞ and $+\infty$ mean the same thing ("positive infinity"); the $+\infty$ is often used for contrast with the $-\infty$ symbol ("negative infinity").

- The formal definitions for infinite limits are the following:
  - We say that a function $f(x)$ diverges to $+\infty$ as $x \to c$, written as $\lim_{x \to c} f(x) = +\infty$, if the following statement is true: for any $B > 0$ (no matter how large) there exists a $\delta > 0$ (depending on B) with the property that for all $0 < |x - c| < \delta$, we have $f(x) > B$.
    - The idea of diverging to $-\infty$ is analogous, except instead $f(x) < -B$.
    - We can also formulate one-sided limits here, with $x \to c$ from above or from below.
  - We say that a function $f(x)$ diverges to $+\infty$ as $x \to \infty$, written as $\lim_{x \to \infty} f(x) = +\infty$, if the following statement is true: for any $B > 0$ (no matter how large) there exists an $A > 0$ (depending on B) with the property that for all $x > A$, we have $f(x) > B$.
    - The statements for diverging to $-\infty$, or the statements as $x \to -\infty$, are similar.

- Here are some sketch-examples of infinite limits evaluated using the definitions:
  - Example: The function $f(x) = x^n$ (for n a positive integer) diverges to $+\infty$ as $x \to +\infty$ and, as $x \to -\infty$, it diverges to $+\infty$ if n is even and to $-\infty$ if n is odd.
    - For this function, we can take $A = \sqrt[n]{B}$, for either direction.
  - Example: The function $f(x) = e^x$ diverges to $+\infty$ as $x \to +\infty$ and tends to 0 as $x \to -\infty$.
    - As $x \to +\infty$ we can take $A = \ln(B)$. As $x \to -\infty$ we can take $A = -\ln(\epsilon)$.
  - Example: The function $f(x) = \dfrac{1}{x^2}$ diverges to $+\infty$ as $x \to 0$.
    - For this function, we can take $\delta = \sqrt{B}$.
  - Example: The function $f(x) = \dfrac{1}{x}$ diverges to $-\infty$ as $x \to 0$ from below, and diverges to $+\infty$ as $x \to 0$ from above.
    - For this function, we can take $\delta = B$, for either direction.

0.0.4 Proofs of the Limit Rules for Infinite Limits

- Basic Limits: $\lim_{x \to +\infty} c = c$ for any constant c, $\lim_{x \to +\infty} \dfrac{1}{x} = 0$, and $\lim_{x \to \infty} x = \infty$.
  - Proof (c): Suppose we are given $\epsilon > 0$. We want to find an $A > 0$ such that for all $x > A$, we have that $|c - c| < \epsilon$. Any value of A will work, since $|c - c| = 0$ is always less than ϵ (which is positive).
  - Proof ($\frac{1}{x}$): Suppose we are given $\epsilon > 0$. We want to find an $A > 0$ such that for all $x > A$, we have that $\left|\dfrac{1}{x}\right| < \epsilon$. We claim that $A = \dfrac{1}{\epsilon}$ will work. To see this, observe that if $x > \dfrac{1}{\epsilon}$, then after multiplying the inequality by $\dfrac{\epsilon}{x}$ we see that $\dfrac{\epsilon}{x} \cdot x > \dfrac{\epsilon}{x} \cdot \dfrac{1}{\epsilon}$, or $\epsilon > \dfrac{1}{x}$. Since $\dfrac{1}{x}$ is positive, we thus obtain $\left|\dfrac{1}{x}\right| < \epsilon$, as desired.
  - Proof (x): We need to show that for any $B > 0$ (no matter how large) there exists an $A > 0$ (depending on B) with the property that for all $x > A$, we have $f(x) > B$. Here, we can simply take $A = B$, since $f(x) = x$.

- Negation: If $\lim_{x \to a} f(x) = \infty$, then $\lim_{x \to a} [-f(x)] = -\infty$, and vice versa.
  - Proof: We need only multiply the inequality for $f(x)$ by -1 to get the necessary inequality for $-f(x)$ (i.e., the values of δ are the same).

- <u>Multiplication</u>: If $\lim_{x\to a} f(x)$ is a finite positive number or ∞ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} [f(x)g(x)] = \infty$.
  - <u>Proof</u>: By the hypothesis about f, there exists some C such that for all $x > C$ it is true that $f(x) > M$ for some positive real number M.
  - Now let $B > 0$ be given. By the hypothesis about g, there exists a $\delta_2 > 0$ such that for all $0 < |x - a| < \delta_2$, we have $g(x) > B/M$.
  - Then for all $0 < |x - a| < \delta$ with $\delta = \min(\delta_1, \delta_2)$, we see that $f(x)g(x) > B$, as desired.

- <u>Addition</u>: If $\lim_{x\to a} f(x)$ is a finite number or ∞ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} [f(x) + g(x)] = \infty$.
  - <u>Proof</u>: By the hypothesis about f, there exists some δ_1 such that for all x with $0 < |x - a| < \delta_1$ it is true that $f(x) > M$ for some (possibly negative) real number M.
  - Now let $B > 0$ be given. By the hypothesis about g, there exists a $\delta_2 > 0$ such that for all $0 < |x - a| < \delta_2$, we have $g(x) > B - M$.
  - Then for all $0 < |x - a| < \delta$ with $\delta = \min(\delta_1, \delta_2)$, we see that $f(x) + g(x) > B$, as desired.

- <u>Division</u>: If $\lim_{x\to a} f(x)$ is a finite number and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = 0$.
  - <u>Proof</u>: By the hypothesis about f, there exists some δ_1 such that for all x with $0 < |x - a| < \delta_1$ it is true that $|f(x)| < M$ for some (nonnegative) real number M.
  - Now let $\epsilon > 0$ be given. By the hypothesis about g, there exists a $\delta_2 > 0$ such that for all $0 < |x - a| < \delta_2$, we have $g(x) > M/\epsilon$.
  - Then for all $0 < |x - a| < \delta$ with $\delta = \min(\delta_1, \delta_2)$, we see that $\frac{f(x)}{g(x)} < \epsilon$, as desired.

- <u>Exponentiation</u>: If $\lim_{x\to a} f(x) = L$ (where $L \geq 0$) and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)^{g(x)}$ is 0 if $0 < L < 1$, and is ∞ if $L > 1$. Furthermore, $\lim_{x\to a} g(x)^{f(x)}$ is ∞ provided $L > 0$.
  - <u>Proof</u> (f^g for $0 < L < 1$): By the hypothesis about f, there exists some δ_1 such that for all x with $0 < |x - a| < \delta_1$ it is true that $|f(x)| < M$ for some real number M with $0 < M < 1$. Now let $\epsilon > 0$ be given: by the hypothesis about g, there exists a $\delta_2 > 0$ such that for all $0 < |x - a| < \delta_2$, we have $g(x) > -\log_M \epsilon$. Then for all $0 < |x - a| < \delta$ with $\delta = \min(\delta_1, \delta_2)$, we see that $f(x)^{g(x)} < \epsilon$, as desired.
  - <u>Proof</u> (f^g for $L > 1$): By the hypothesis about f, there exists some δ_1 such that for all x with $0 < |x - a| < \delta_1$ it is true that $|f(x)| > M$ for some real number M with $M > 1$. Now let $B > 0$ be given: by the hypothesis about g, there exists a $\delta_2 > 0$ such that for all $0 < |x - a| < \delta_2$, we have $g(x) > \log_M B$. Then for all $0 < |x - a| < \delta$ with $\delta = \min(\delta_1, \delta_2)$, we see that $f(x)^{g(x)} > B$, as desired.
  - <u>Proof</u> (g^f): By the hypothesis about f, there exists some δ_1 such that for all x with $0 < |x - a| < \delta_1$ it is true that $|f(x)| > M$ for some positive real number M. Now let $B > 0$ be given: by the hypothesis about g, there exists a $\delta_2 > 0$ such that for all $0 < |x - a| < \delta_2$, we have $g(x) > B^{1/M}$. Then for all $0 < |x - a| < \delta$ with $\delta = \min(\delta_1, \delta_2)$, we see that $g(x)^{f(x)} > B$, as desired.

Well, you're at the end of my handout. Hope it was helpful.